The Hamiltonian formulation of General Relativity: myths and reality

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Abstract

A conventional wisdom often perpetuated in the literature states that: (i) a 3+1 decomposition of space-time into space and time is synonymous with the canonical treatment and this decomposition is essential for any Hamiltonian formulation of General Relativity (GR); (ii) the canonical treatment unavoidably breaks the symmetry between space and time in GR and the resulting algebra of constraints is not the algebra of four-dimensional diffeomorphism; (iii) according to some authors this algebra allows one to derive only spatial diffeomorphism or, according to others, a specific field-dependent and non-covariant four-dimensional diffeomorphism; (iv) the analyses of Dirac [*Proc. Roy. Soc. A 246 (1958) 333*] and of ADM [*Arnowitt, Deser and Misner, in "Gravitation: An Introduction to Current Research" (1962) 227*] of the canonical structure of GR are equivalent. We provide some general reasons why these statements should be questioned. Points (i-iii) have been shown to be incorrect in [*Kiriushcheva et al., Phys. Lett. A 372 (2008) 5101*] and now we thoroughly re-examine all steps of the Dirac Hamiltonian formulation of GR. By direct calculation we show that Dirac's references to space-like surfaces are inessential and that such surfaces do not enter his calculations. In addition, we show that his assumption $g_{0k} = 0$, used to simplify his calculation of different contributions to the secondary constraints, is unwarranted; yet, remarkably his total Hamiltonian is equivalent to the one computed without the assumption $g_{0k} = 0$. The secondary constraints resulting from the conservation of the primary constraints of Dirac are in fact different from the original constraints that Dirac called secondary (also known as the "Hamiltonian" and "diffeomorphism" constraints). The Dirac constraints are instead particular combinations of the constraints which follow from the primary constraints. Taking this difference into account we found, using two standard methods, the generator of the gauge transformation gives diffeomorphism invariance in four-dimensional space-time; and this shows that points (i-iii) above cannot be attributed to the Dirac Hamiltonian formulation of GR. We also demonstrate that ADM and Dirac formulations are related by a transformation of phase-space variables from the metric $g_{\mu\nu}$ to lapse and shift functions and the three-metric g_{km}, which is not canonical. This proves that point (iv) is incorrect. Points (i-iii) are mere consequences of using a non-canonical change of variables and are not an intrinsic property of either the Hamilton-Dirac approach to constrained systems or Einstein's theory itself.

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"On ne trouvera point de Figures dans cet Ouvrage. Les méthodes que j'y expose ne demandent ni constructions, ni raisonnemens géométriques ou mécaniques, mais seulement des opérations algébriques, assujéties à une marche régulière et uniforme. Ceux qui aiment ľ Analyse, verront avec plaisir la Mécanique en devenir une nouvelle branche, et me sauront gré ď en avoir étendu ainsi le domaine."

J. L. Lagrange, "Mécanique Analytique" (1788)

The reader will find no figures in this work. The methods which I set forth do not require either constructions or geometrical or mechanical reasonings, but merely algebraic operations subjected to a regular and uniform rule of procedure. Those who are fond of Mathematical Analysis will observe with pleasure Mechanics becoming one of its new branches and they will be grateful to me for having thus extended its domain.

I. INTRODUCTION

We begin our paper with words written more than two centuries ago by Lagrange in the preface to the first edition of the "Mécanique Analytique" [1] because they express our standpoint in analyzing of the Hamiltonian formulation of General Relativity (GR). The results previously obtained by others are reconsidered and classified as either "myth" or "reality" depending on whether they were obtained by what Lagrange called *a regular and uniform rule of procedure,* or by geometrical or some other reasonings. The results and conclusions constructed using such reasonings must be checked by explicit calculation; without which they are meaningless and could be misleading, contradicting the rules of procedure and the essential properties of GR.

Originating more than half a century ago, the Hamiltonian formulation of GR is not a

new subject. It began with advances in the Hamiltonian formulation of singular Lagrangians due to the pioneering work of Dirac on generalized (constrained) Hamiltonian dynamics [2].

We restrict our discussion to the original Einstein metric formulation of GR. The first-order, metric-affine, form [3] will be just briefly touched; but the analysis presented here can and must be extended to a metric-affine form and to other formulations.

In chronological order (which is also ranked inversely in popularity) the Hamiltonian formulation of GR was considered by Pirani, Schild, and Skinner (PSS) [4], Dirac [5], and Arnowitt, Deser, and Misner (ADM) [6] and references therein. The relationship among these formulations has not been analyzed; and some authors have adopted to using the name "Dirac-ADM" or refer to Dirac when actually working with the ADM Hamiltonian. This presumes equivalence of the Dirac and ADM formulations. These two, as we will demonstrate, are not equivalent.

The Dirac conjecture [7], that knowing all the first-class constraints is sufficient to deduce the gauge transformations, was made only after the appearance of [4, 5, 6] and became a well defined procedure only later [8, 9]. The application of such a procedure to field theories was considered for the first time by Castellani [10] (for alternative approaches see [11, 12, 13]). Deriving the gauge invariance of GR from the complete set of the first-class constraints should also be viewed as a crucial consistency condition that must be met by any Hamiltonian formulation of the theory; yet, this requirement did not attract much attention and it is not discussed in textbooks on GR, where a Hamiltonian formulation is presented (e.g. [14, 15]). In books on constraint dynamics [16, 17, 18], even if such a procedure is discussed [18], it is not applied to the Hamiltonian formulation of GR. Recently this question was again brought to light by Mukherjee and Saha [19] who applied the method of [12] to the ADM Hamiltonian with the sole emphasis on presenting the method of deriving the gauge invariance, not on the results themselves. In [19] there appears a first complete derivation of the gauge transformations from the constraint structure of the ADM Hamiltonian. The expected transformation of the metric tensor is [20]

$$\delta g_{\mu\nu} = -\xi_{\mu;\nu} - \xi_{\nu;\mu}, \tag{1}$$

where ξ_μ is the gauge parameter and the semicolon ";" signifies the covariant derivative. In the literature on the Hamiltonian formulation of GR, the word "diffeomorphism" is often used as equivalent to the transformation (1), which is similar to gauge transformations in

ordinary field theories. This meaning is employed in our article.[1] The expected invariance (1) does not follow from the constraint structure of ADM Hamiltonian and a *field-dependent* and *non-covariant* redefinition of gauge parameters is needed[2] to present the transformations of [19] in the *form* of (1), i.e.

$$\xi^0 = \left(-g^{00}\right)^{1/2} \varepsilon^{\perp}_{ADM}, \quad \xi^k = \varepsilon^k_{ADM} + \frac{g^{0k}}{g^{00}} \left(-g^{00}\right)^{1/2} \varepsilon^{\perp}_{ADM}. \tag{2}$$

The field-dependent redefinition of gauge parameters (2) goes back to work of Bergmann and Komar [21] where it was presented for the first time. The same redefinition of gauge parameters (2), but in a less transparent form, was obtained for the ADM Hamiltonian by Castellani [10] for the transformation of the $g_{0\mu}$ components of the metric tensor to illustrate his procedure for the construction of the gauge generators. This redefinition of gauge parameters was also discussed from different points of view in [22, 23, 24, 25], the most recent derivation is in [19]. A common feature of these different approaches is that they only consider the ADM Hamiltonian. According to the conclusion of [21], the transformation (1) and the one with parameters that depend on the fields (2) are *distinct*. In [23] this transformation is called the "specific metric-dependent diffeomorphism". The authors of [19] have a brief and ambiguous conclusion about (2): "[it will] lead to the *equivalence*[3] between the diffeomorphism and gauge transformations" and, at the same time, "demonstrate the unity of the *different* symmetries involved"; these are contradictory statements.

Soon after appearance of [19], Samanta [26] posed the question "whether it is possible to describe the diffeomorphism symmetries without recourse to the ADM decomposition". To answer this question, he derived the transformation (1) starting from the Einstein-Hilbert (EH) Lagrangian (not the ADM Lagrangian) and applying the Lagrangian method for recovering gauge symmetries based on the use of certain gauge identities that appear in [17]. It is important that (1) follows exactly from this procedure without the need of field-dependent and non-covariant redefinition of the gauge parameters, which would be necessary in [10, 19] where the ADM Hamiltonian is used. The question of the equivalence of (1) and (2) does

[1] In mathematical literature the term diffeomorphism refers to a mapping from one manifold to another which is differentiable, one-to-one, onto, with a differentiable inverse.

[2] More detail on the derivation of (2) is given in the last Section where application of Castellani's procedure to the ADM Hamiltonian is reexamined ($\varepsilon^{\perp}_{ADM}$ and ε^{k}_{ADM} are gauge parameters of ADM formulation).

[3] Here and everywhere in this article the *Italic* in quotations is ours.

not even arise in the approach of [26]. In [26] the diffeomorphism transformations were also derived by applying the same method to the first-order, affine-metric, formulation [3] of GR. The conclusion of [26] that "the ADM splitting, which is *essential* for discussing diffeomorphism symmetries, is *bypassed*" contradicts the obtained result. Firstly, any feature that is "*essential*" cannot be "*bypassed*". Secondly, the transformations derived from the ADM Hamiltonian in [19] are not those of [26]. It is not a "bypass" because the "destination" of having the invariance of (1) is changed.

The conclusion about the results of [19] and [26] should be that the ADM decomposition is inessential and incorrect because it does not lead to diffeomorphism invariance. This discrepancy between these two recent results vindicates Hawking's old statement [27] "the split into three spatial dimensions and one time dimension seems to be contrary to the whole spirit of relativity", the more recent statements of Pons [24]: "Being non-intrinsic, the 3+1 decomposition is somewhat at odds with a generally covariant formalism, and difficulties arise for this reason", and Rovelli [28]: "The very foundation of general covariant physics is the idea that the notion of a simultaneity surface all over the universe is devoid of physical meaning".

There is another statement in [26] that can also be found in many places "it is well known that this decomposition plays a central role in *all Hamiltonian formulations* of general relativity". This sentence combined with Hawking's "spiritual" statement forces one to conclude that the Hamiltonian formulation by itself contradicts the spirit of GR. This resonates with Pullin's conclusion [29] that "Unfortunately, the canonical treatment breaks the symmetry between space and time in general relativity and the resulting algebra of constraints is not the algebra of *four diffeomorphism*". We will show in this paper that the canonical formalism is in fact consistent with the diffeomorphism (1) when the Dirac constraint formalism is applied consistently and that the discrepancies between the ADM formalism and (1) can be explained.

The difference of the results [19] and [26] which were obtained by different methods also implies the non-equivalence of the Lagrangian and Hamiltonian formulations. In all field theories (e.g., Maxwell or Yang-Mills) the Hamiltonian and Lagrangian formulations give the same result for gauge invariance, so for GR to differ seems unnatural. Could this be a peculiar property of GR? Is GR a theory in which the Hamiltonian and Lagrangian formulations lead to different results or was a "rule of procedure" broken somewhere?

Recently, in collaboration with Racknor and Valluri [30], we demonstrated that, by following the most natural first attempt of PSS [4] and by applying the rules of procedure [2, 7, 10, 16, 17, 18], the Hamiltonian formulation of GR (without any modifications of the action or change of variables) leads to consistent results. The gauge transformation of the metric tensor was derived using the method of [10] and, without any field-dependent redefinitions of gauge parameters, it gives exactly the same result as the Lagrangian approach of [26], as it should. In the Hamiltonian formulation of GR given in [30] the algebra of constraints is the algebra of "*four diffeomorphism*", in contradiction to the *general* conclusion of [29] which was based on the *particular*, ADM, formulation.

The procedure of passing to a Hamiltonian formulation in field theories based on the separation of the space and time *components* of the fields and their *derivatives* (defined on the whole space-time, not on some hypersurface) is not equivalent to separation of space-time into space and time. For example, by rewriting the Einstein equations in components (as was done before Einstein introduced his condensed notation), we do not abandon covariance even if it is not manifest. In addition, such explicit separation of the space and time *components* and the *derivatives* of the fields does not affect space-time itself and is not to be associated with any 3+1 decomposition, slicing, splitting, foliation, etc. of space-time. The final result for the gauge transformation of the fields can be presented in covariant form when using the Hamiltonian formulation of ordinary field theories (e.g., Yang-Mills, Maxwell), as well as in GR [30]. In any field theory, after rewriting its Lagrangian in components, the Hamiltonian formulation for singular Lagrangians follows a well defined procedure. Such a procedure is based on consequent calculations of the Poisson brackets (PB) of constraints with the Hamiltonian using the fundamental PBs of independent fields. In the case of field theories they are

$$\left\{q\left(x^0,\mathbf{x}\right),p\left(y^0,\mathbf{y}\right)\right\}_{x^0=y^0}=\delta\left(\mathbf{x}-\mathbf{y}\right). \tag{3}$$

This is a local relation that does not rely on any extended objects or surfaces. Again, as with separation into components, this locally defined canonical PB does not affect space-time and is not related to space-like surface or any other hypersurface because (3) is zero for $\mathbf{x}\neq\mathbf{y}$ in a whole space-time and there is no information in (3) that, using mathematical language, can allow one to classify two separate points as points on a particular space-like surface or on any surface. The canonical procedure does not itself lead to the appearance of

any hypersurfaces; in [30] there are no references to such surfaces and the result is consistent with the Lagrangian formulation of [26]. Such surfaces are either a phantom of interpretation or canonical procedure was abandoned by their introduction.

The discussion of an interpretational approach is not on the main road of our analysis of the Hamiltonian formulations of GR. However, the routes of such an approach[4] are quite interesting: one starting from the basic equations of the ADM formulation, according to [31], "would like to understand intuitively their geometrical and physical meaning and derive them from some first principles rather than by a formal rearrangement of Einstein's law". By taking this approach, a formal rearrangement (which is a "rule of procedure") is replaced by some sort of intuitive understanding. As a result, a new language is created which "is much closer to the language of quantum dynamics than the original language of Einstein's law ever was" [31]. This language allows one "to recover the old comforts of a Hamiltonian-like scheme: a system of hypersurfaces stacked in a well defined way in space-time, with the system of dynamic variables distributed over these hypersurfaces and developing uniquely from one hypersurface to another" [32]. Such an interpretation, although 'reasonable' from the point of view of classical Laplacian determinism, is hard to justify from the standpoint of GR [33]. In GR, an entire spatial slice can only be seen by an observer in the infinite future [34] and an observer at any point on a space-like surface does not have access to information about the rest of the surface (this is reflected in the local nature of (3) in field theories). It would be non-physical to build any formalism by basing it on the development in time of data that can be available only in the infinite future and trying to fit GR into a scheme of classical determinism and nonrelativistic Quantum Mechanics with its notion of a wave function defined on a space-like slice. The condition that a space-like surface remains space-like obviously imposes restrictions on possible coordinate transformations, thereby destroying four-dimensional symmetry, and, according to Hawking, "it restricts the topology of space-time to be the product of the real line with some three-dimensional manifold, whereas one would expect that quantum gravity would allow all possible topologies of space-time including those which are not product" [27]. This restriction, imposed by the slicing of space-time, must be lifted at the quantum level [35]; but, from our point of view,

[4] We have to confess that we found hard to understand approaches which are not analytical and, to avoid any misinterpretations, we will merely quote their advocates. A reader interested in this approach can find more details in the articles we cite.

avoiding it at the outset seems to be the most natural cure for this problem.

The usual interpretation of the ADM variables, constraints, and Hamiltonian obviously contradicts the spirit of relativity. With restrictions on coordinate transformations which are imposed by such an interpretation it is quite natural to expect something different from a diffeomorphism transformation, as was found in [10, 19].

Any interpretation, whether or not it contradicts the spirit of GR, cannot provide a sufficiently strong argument to prove or disprove some particular result or theory, because an arbitrary interpretation cannot change or affect the result of formal rearrangements. The transformation different from diffeomorphism that follows from the ADM Hamiltonian is the result of a definite procedure [10, 19] and is based on calculations performed with their variables and their algebra of constraints. From the beginning we will not use the language of 3+1 dimensions, so as to avoid the necessity of getting ourselves "out of space and back into space-time" [36] at the end of the calculations. In any case, it would likely be impossible to do so after we have gone beyond the point of no return on such a road. We must reexamine the *derivation* of ADM Hamiltonian right from the start.

It is difficult to compare the results of [30] directly with those of ADM because some additional modifications of the original GR Lagrangian were performed by ADM and it is not easy to trace them according to the "rules of procedure". We will start with the work of Dirac [5], where all modifications and assumptions are explicitly stated making it possible for them to be checked and analyzed. In addition, Dirac's canonical variables are components of the metric tensor which are the same as those used in [30] where diffeomorphism invariance was derived directly from the Hamiltonian and constraints. Moreover, in [37] two Hamiltonian formulations, based on the linearized Lagrangians of [4, 30] and [5], were considered. Despite there being different expressions for the primary and secondary constraints, these two formulations have the same algebra of PBs among the constraints, and with the Hamiltonian, therefore, they have the same gauge invariance. This is exactly what one can expect in the case of full GR, provided one makes no deviation from canonical procedure. In analyzing the ADM formulation we will follow a different path. We will not start from the GR Lagrangian, but instead compare the final results of Dirac and ADM and try to determine what deviation from the canonical procedure lead to the transformations found in [10, 19] which are distinct from those of (1).

In the next Section we shall thoroughly reexamine the Dirac derivation of the GR Hamil-

tonian [5] with emphasis on the effect of his modifications of the action and of the other simplifying assumptions he makes. In particular, we will investigate whether space-like surfaces actually play any role in his derivation, or if they just serve as an illustration which can be completely disregarded from the standpoint of the canonical procedure, as in [30]. In Section III, using Castellani's procedure and the results of Section II, we derive the transformations of the metric tensor. The result is the same as those found in [26] and [30]. The same result is obtained by application of the method used in [19] to Dirac's Hamiltonian, which illustrates the equivalence of these two methods. Some peculiarities of such methods, that cannot be seen in ordinary field theories, are briefly discussed and related to the peculiarities of diffeomorphism invariance as it compares to the gauge invariance in ordinary theories. Finally, we consider the ADM Hamiltonian formulation of GR. In the last Section IV we demonstrate that the ADM formulation follows from Dirac's by a change of variables. The canonicity of this change of variables (the ADM lapse and shift functions) is analyzed. Based on this analysis, the general and more restrictive criteria for a canonical transformation in the case of singular gauge invariant theories are discussed.

II. ANALYSIS OF DIRAC DERIVATION

In [30] the GR Hamiltonian, constraints, closure of the Dirac procedure, and the diffeomorphism transformation of the metric tensor were derived without any reference to space-like surfaces, the use of any 3+1 decomposition of space-time, or slicing, splitting, foliation, etc., as well as without modifications of the Lagrangian or the introduction of any new variables. (The canonical variables of [30] are components of the metric tensor.) Dirac, when considering the Hamiltonian formulation of GR in [5], also used the metric tensor as a canonical variable; but he made frequent references to space-like surfaces. If such surfaces, which according to Hawking contradict the whole spirit of General Relativity, are the part of Dirac's calculations, then one has to expect transformations different from diffeomorphism and similar to the one found in [19] from the ADM Hamiltonian. Our main interest is to find out, by following all the steps of Dirac's derivation of the Hamiltonian, the place where (if anywhere) space-like surfaces enter his *derivation* or where (if anywhere) his approach *deviates* from a regular and uniform rule of canonical procedure. If there is no deviation, one should then obtain the diffeomorphism invariance (1), the same as found in [30]. This

would resemble what happens in linearized GR, as discussed in [37].

In [5], Dirac started the Hamiltonian formulation from the "gamma-gamma" part of the Einstein-Hilbert (EH) Lagrangian (Eq. (D8))[5] (e.g., see [20, 38])

$$L_G = \sqrt{-g}g^{\mu\nu}\left(\Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\nu\sigma}\right) = \frac{1}{4}\sqrt{-g}g_{\mu\nu,\rho}g_{\alpha\beta,\sigma}B^{\mu\nu\rho\alpha\beta\sigma} \tag{4}$$

where

$$B^{\mu\nu\rho\alpha\beta\sigma} = \left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}\right)g^{\rho\sigma} + 2\left(g^{\mu\rho}g^{\alpha\beta} - g^{\mu\alpha}g^{\beta\rho}\right)g^{\nu\sigma}. \tag{5}$$

The same Lagrangian was used in [4] and [30]. This is a Lagrangian of a local field theory in four(or any)-dimensional space-time, and space-like surfaces or any other hypersurfaces are not intrinsic to such a formulation.

The primary constraints (the ϕ-equations of [5]) that follow from (4) are

$$\phi^{\mu 0} = p^{\mu 0} - \frac{\delta L_G}{\delta g_{\mu 0,0}} \approx 0, \tag{6}$$

where $p^{\mu\nu}$ are momenta conjugate to $g_{\mu\nu}$. The exact form of $\phi^{\mu 0}$ can be found in [4, 30] (Greek subscripts run from 0 to $d-1$ and Latin ones from 1 to $d-1$ where d is the dimension of space-time).

In addition to eliminating the second order derivatives of the metric tensor present in the Ricci scalar in passing from the EH Lagrangian to its gamma-gamma part (4) so that [38]

$$L_{EH} = \sqrt{-g}R = L_G + \partial_\mu V^\mu, \tag{7}$$

Dirac made an additional change to the Lagrangian in order to eliminate the second term in (6). The modified Lagrangian is obtained by adding two total derivatives which are non-covariant (Eq. (D15))

$$L^* = L_G + \left[\left(\sqrt{-g}g^{00}\right)_{,v}\frac{g^{v0}}{g^{00}}\right]_{,0} - \left[\left(\sqrt{-g}g^{00}\right)_{,0}\frac{g^{v0}}{g^{00}}\right]_{,v}. \tag{8}$$

[5] We will refer on Dirac equations quite often and use the convention, Eq. (D#), to mean equation # from [5].

This change does not affect the equations of motion, but leads to simple primary constraints (Eq. (D14))

$$\phi^{\mu 0} = p^{\mu 0} \approx 0. \tag{9}$$

It was shown in [37], that the linearized version of the modified (8) and unmodified Lagrangians (4), despite leading to different expressions for the constraints and the Hamiltonian, result in the same constraint structure, the same number of first-class constraints, and the same gauge invariance, which is the linearized version of diffeomorphism. This is what one can also expect in the case of full GR. According to [5], the simplification (9) "can be achieved only at the expense of abandoning four-dimensional symmetry" which is obviously correct for this modification of the Lagrangian (8); yet Dirac's further conclusion that "four-dimensional symmetry is not a fundamental property of the physical world" is too strong and has to be clarified. Of course, four-dimensional symmetry of the Lagrangian is destroyed by the modification (8); but this change does not affect the equations of motion, which are the same as the Einstein equations. Consequently, for the equations of motion, not only four-dimensional symmetry is preserved, but also general covariance.[6] If four-dimensional symmetry is preserved in the equations of motion, which are invariant under general coordinate transformations, then diffeomorphism should be recovered in the course of the Hamiltonian analysis, as in [30].

The new Lagrangian L^* differs from the original one (4) only for terms linear in the time derivatives of a metric (i.e. 'velocities'), the parts responsible for the simplification of the primary constraints. We then have

$$L^* = L_G\left(2\right) + L^*\left(1\right) + L_G\left(0\right), \tag{10}$$

where the numbers in brackets indicate the order in velocities (for the Hamiltonian and constraints it will indicate the order in momenta). The exact form of $L^*\left(1\right)$ is given by Eq. (D18).

This Lagrangian is used to pass to the Hamiltonian

[6] The term "four-dimensional" symmetry used by Dirac probably reflects the fact that the gamma-gamma part of the Lagrangian, quadratic in first order derivatives, is not generally covariant after the elimination of terms with second order derivatives in the full EH Lagrangian (7).

$$H = g_{\alpha\beta,0} p^{\alpha\beta} - L^*. \tag{11}$$

With the modification of (8) the part of the Lagrangian $L_G(2) + L^*(1)$, as was shown by Dirac, can be written as

$$L_G(2) + L^*(1) = L_X(0) - \sqrt{-g}\frac{1}{g^{00}} E^{rsab} \Gamma^0_{rs} \Gamma^0_{ab} \tag{12}$$

where $\Gamma^\mu_{\alpha\beta}$ is the Christoffel symbol

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu} \right) \tag{13}$$

and

$$E^{rsab} = e^{rs} e^{ab} - e^{ra} e^{sb} \tag{14}$$

with

$$e^{\alpha\beta} = g^{\alpha\beta} - \frac{g^{0\alpha} g^{0\beta}}{g^{00}}. \tag{15}$$

Note, that in the second order formulation, $\Gamma^\mu_{\alpha\beta}$, $E^{\alpha\beta\mu\nu}$, and $e^{\alpha\beta}$ are just short notations and none of them denote a new and/or independent variable.

Some comments about (12) are in order. The careful reader will definitely wonder how the parts of the Lagrangian which are quadratic and linear in velocities can have contributions without velocities, $L_X(0)$; the direct calculation of $L_G(2) + L^*(1)$ does not have such contributions (see Dirac's unnumbered equation preceding (D19))

$$\frac{1}{4}\sqrt{-g} E^{rasb} \left[g_{rs,0} g_{ab,0} g^{00} + 2 g_{rs,0} g_{ab,v} g^{v0} - 4 g_{rs,0} g_{a\beta,b} g^{\beta 0} \right]. \tag{16}$$

Dirac completed this square, leading to the compact form of (12). Working with (16) instead of (12), will of course not change the results and actually has no calculational advantage. However, we keep (12) so as to compare our calculations with those of Dirac.

The $L_X(0)$ in (12) (explicitly given by (24)) is independent of the velocities. The only part of (11) that has dependence on $g_{rs,0}$ is

$$g_{rs,0} p^{rs} + \sqrt{-g} \frac{1}{g^{00}} E^{rsab} \Gamma^0_{rs} \Gamma^0_{ab}. \tag{17}$$

Performing the variation $\frac{\delta}{\delta g_{rs,0}}$, we obtain (see (D18-D21))

$$p^{rs} = \sqrt{-g} E^{rsab} \Gamma^0_{ab} = \frac{1}{2} \sqrt{-g} E^{rsab} \left[g^{00} \left(g_{a0,b} + g_{b0,a} - g_{ab,0} \right) + g^{0k} \left(g_{ak,b} + g_{bk,a} - g_{ab,k} \right) \right] . \tag{18}$$

Equation (18) is easy to solve for $g_{ab,0}$ due to the invertability of E^{rsab}

$$E^{rsab} I_{abmn} = \delta^r_m \delta^s_n, \tag{19}$$

where the inverse to E^{rsab} in any space-time dimension d (except $d = 2$) is

$$I_{abmn} = \frac{1}{d-2} g_{ab} g_{mn} - g_{am} g_{bn}. \tag{20}$$

This result gives

$$g_{mn,0} = -2 \frac{1}{\sqrt{-g}} \frac{1}{g^{00}} p^{rs} I_{rsmn} + g_{m0,n} + g_{n0,m} + \frac{g^{0k}}{g^{00}} \left(g_{mk,n} + g_{nk,m} - g_{mn,k} \right) . \tag{21}$$

After substitution of (21) into (17) (note that (18) can be solved for Γ^0_{ab} thus making the calculations shorter) we obtain the total Hamiltonian

$$H_T = g_{00,0} p^{00} + 2 g_{0k,0} p^{0k} + H_G, \tag{22}$$

where H_G (the canonical part of the Hamiltonian) is given by Eqs. (D33, D34)) as,

$$H_G = -\frac{1}{g^{00}\sqrt{-g}} I_{rsab} p^{rs} p^{ab} + g_{u0} e^{uv} \left[p^{rs} g_{rs,v} - 2 \left(p^{rs} g_{rv} \right)_{,s} \right] - L_X(0) - L_G(0) , \tag{23}$$

with $L_X(0)$ (Eq. (D19)) and $L_G(0)$ (Eq. (D8)):

$$L_X(0) = \frac{1}{4} \frac{\sqrt{-g}}{g^{00}} E^{rsab} \left[g_{rs,u} g^{u0} - \left(g_{r\alpha,s} + g_{s\alpha,r} \right) g^{\alpha 0} \right] \left[g_{ab,v} g^{v0} - \left(g_{a\beta,b} + g_{b\beta,a} \right) g^{\beta 0} \right] , \tag{24}$$

$$L_G(0) = \frac{1}{4} \sqrt{-g} g_{\mu\nu,k} g_{\alpha\beta,t} B^{\mu\nu k\alpha\beta t}. \tag{25}$$

Note that the second term of (23), the part linear in the momenta, arises only after some rearrangement. The direct substitution of $g_{rs,0}$ into $g_{rs,0} p^{rs}$ (the only part of (11) that leads to terms linear in the momenta) gives

$$2p^{mn}g_{m0,n}+\frac{g^{0k}}{g^{00}}p^{mn}\left(2g_{mk,n}-g_{mn,k}\right), \tag{26}$$

which after integration by parts and using $\frac{g^{0k}}{g^{00}}=-g_{0m}e^{mk}$ leads to

$$-2g_{m0}\left[p^{mn}_{,n}+e^{mk}p^{mn}\left(g_{mk,n}-\frac{1}{2}g_{mn,k}\right)\right]+2\left(p^{mn}g_{m0}\right)_{,n}. \tag{27}$$

The first term of (27) can be written in the form given by Dirac (D41)

$$-2g_{m0}\left[p^{mn}_{,n}+e^{mk}p^{mn}\left(g_{mk,n}-\frac{1}{2}g_{mn,k}\right)\right]=g_{m0}e^{mv}\mathcal{H}_v, \tag{28}$$

with

$$\mathcal{H}_v=p^{rs}g_{rs,v}-2\left(p^{rs}g_{rv}\right)_{,s}. \tag{29}$$

We note that in obtaining the expression for the Hamiltonian (23), all direct calculations with the initially modified Lagrangian (12) were performed by Dirac without any reference to space-like surfaces or any additional restrictions or assumptions.

The next step in the canonical procedure is to find the time development of the primary constraints and see if there are any secondary constraints (or χ-equations in Dirac's terminology). PBs among the primary constraints are obviously zero, $\left\{p^{0\alpha},p^{0\beta}\right\}=0$. The PBs of the primary constraints (9) with the total Hamiltonian (22) are

$$\left\{p^{0\sigma},H_T\right\}=\frac{\delta}{\delta g_{0\sigma}}H_G=\chi^{0\sigma}, \tag{30}$$

where we keep Dirac's convention for the fundamental PB (Eq. (D11)),

$$\left\{p^{\alpha\beta}\left(x\right),g_{\mu\nu}\left(x'\right)\right\}=\frac{1}{2}\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}+\delta^{\beta}_{\mu}\delta^{\alpha}_{\nu}\right)\delta_3\left(x-x'\right). \tag{31}$$

According to Dirac, "the second term of (D33) [$H_G\left(0\right)=-L_X\left(0\right)-L_G\left(0\right)$ in our (23)] is very complicated and a great deal of labour would be needed to calculate it directly" and instead of performing the variation $\frac{\delta}{\delta g_{0\sigma}}H_G\left(0\right)$, he uses some arguments (see (D23-D27)) related to the displacements of surfaces of constant time, and thus he infers that the Hamiltonian "must be of the form" (see (D28))

$$H=\left(g^{00}\right)^{-\frac{1}{2}}\mathcal{H}_L+g_{r0}e^{rs}\mathcal{H}_s,$$

where $\mathcal{H}_L$ and $\mathcal{H}_s$ are independent of the $g_{0\mu}$. Dirac's arguments are very general and independent of the particular form of the Lagrangian, i.e. they have no connection with his initial modifications of L_G leading to L^*. And, even in the linearized case [37], without these modifications, the secondary constraints have a dependence on $g_{0\mu}$; this dependence also happens in full GR [30]. In any case, the explicit form of the constraints cannot be found using such arguments and explicit calculations are needed; one has to use a well defined rule of procedure to find them, i.e. we must calculate $\frac{\delta}{\delta g_{0\sigma}} H_G(0)$. Dirac performed these calculations using an additional simplifying assumption (see below) and this result has to be analyzed and compared to what follows from direct calculations.

According to Dirac, there are no contributions from $H_G(0)$ to a vector constraint ($\mathcal{H}^r = e^{rs}\mathcal{H}_s$ in Dirac's notation) which presumably comes from the time development of the corresponding primary constraint $\phi^{r0} = p^{r0}$ (30). Furthermore, $\mathcal{H}_L$, which comes from the time development of the primary constraint, ϕ^{00}, can be calculated with the additional simplifying assumption $g_{r0} = 0$, which gives (Eq. (D36)):

$$g^{r0} = 0, \; g^{rs} = e^{rs}, \; g^{00} = \frac{1}{g_{00}}. \tag{32}$$

As a result, all of $L_X(0)$, along with the biggest part of $L_G(0)$, is dropped from his calculations. According to Dirac [5], the equation for $\mathcal{H}_L$ "must hold also when g_{r0} does not vanish". It is important to check this assumption by direct calculation because if the result of $\frac{\delta}{\delta g_{0\sigma}} H_G(0)$ is the same as that of Dirac's, then the simplifying assumption of (32), along with any references to surfaces of constant time, has nothing to do with his final result. In such a case, Hawking's criticism of formulations based on the introduction of space-like surfaces, which is in contradiction with the whole spirit of General Relativity and restricts topology of space-time [27], cannot be applied to the Dirac analysis of GR. This also means that the transformations (1) should be derivable in the Dirac Hamiltonian formulation, as was done in the Lagrangian formulation [26] or for the Hamiltonian formulation obtained in [30].

If the results following from the assumption of (32) are different from those where the assumption is not made, then we cannot use (32) as an extra condition in the midst of the calculations and we have to go back to the original Lagrangian to introduce this condition from the outset. This is the rule followed in ordinary constraint dynamics; all *imposed* constraints must be solved at the Lagrangian level, or added to the Lagrangian using Lagrange

multipliers, before performing a variation and/or considering the Hamiltonian formulation.

For example, when Chandrasekhar considers the Hamiltonian for Schwarzschild space-time he, first of all, writes the Lagrangian using this metric and only then passes to the Hamiltonian formulation [39]. Similarly, the condition (32) corresponds to a particular coordinate system, one which is *static* [20, 38]; and, of course, the momenta p^{0k}, which are conjugate to the eliminated variables g_{0k} cannot appear in such a formulation. Note that the initial modification of the Lagrangian (8) is irrelevant in a static coordinate system as the last two terms in (8) are zero when g^{0k} is zero. For field theories, especially generally covariant ones, there is an additional restriction: the unambiguous canonical formulation must be performed without explicit reference to ambient space-time by making an *a priori* choice of a particular coordinate system or subclass of coordinate systems [40], i.e. without destroying the main feature of a theory from the beginning.

To find out whether or not Dirac's formulation is correct or any reference to surfaces of constant time and the simplifications of (32) [or (D36) of [5]] are relevant to his actual results, we perform a "great deal of labour" to find the functional derivatives $\frac{\delta}{\delta g_{0\sigma}}$ separately for each contribution of $H_G(0) = -L_X(0) - L_G(0)$ and to compare the results with those obtained by Dirac.

For $L_G(0)$ in (25), we find that

$$\chi_G^{0\sigma}(0) = \left\{p^{0\sigma}, -L_G(0)\right\} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta,kt}\left(g^{0\sigma}E^{\alpha\beta kt} - g^{0t}E^{\alpha\beta k\sigma} - g^{0\beta}E^{tk\sigma\alpha}\right)$$

$$+\frac{1}{4}\sqrt{-g}g_{\mu\nu,k}g_{\alpha\beta,t}\left[C^{\mu\nu k\alpha\beta t}(eee) + C^{\mu\nu k\alpha\beta t}(ee)\right], \tag{33}$$

where the C's are combinations of terms of different order in $e^{\alpha\beta}$ (note that the terms of first and zero orders in $e^{\alpha\beta}$ cancel)

$$C^{\mu\nu k\alpha\beta t}(eee) = g^{0\sigma}\left(-\frac{1}{2}E^{\mu\nu\alpha\beta}e^{kt} + E^{kt\alpha\nu}e^{\mu\beta} + 2E^{\alpha\beta\nu t}e^{\mu k}\right)$$

$$+g^{0k}\left(e^{\beta\mu}E^{\sigma\nu t\alpha} + e^{\sigma\nu}E^{\alpha t\beta\mu}\right) + g^{0\alpha}\left(e^{\mu\nu}E^{\sigma\beta tk} + 2e^{\nu\beta}E^{\sigma t\mu k} - 2e^{\nu t}E^{\sigma\beta\mu k}\right) \tag{34}$$

and

$$C^{\mu\nu k\alpha\beta t}(ee) = \frac{g^{0\alpha}g^{\beta 0}}{g^{00}}\left(E^{tk\mu\sigma}g^{\nu 0} - 2E^{kt\mu\sigma}g^{\nu 0} - E^{\mu\nu\sigma t}g^{k0}\right)$$

$$+\frac{g^{0\sigma}}{g^{00}}\left(-\frac{1}{2}E^{\mu\nu\alpha\beta}g^{k0}g^{t0}+E^{kt\mu\beta}g^{\alpha0}g^{\nu0}+E^{\mu tk\alpha}g^{\beta0}g^{\nu0}+2E^{\alpha\beta k\mu}g^{\nu0}g^{t0}\right). \quad (35)$$

When $\sigma=0$, the result (33), is considerably simplified (this is because $e^{\alpha\beta}$ or $E^{\mu\nu\alpha\beta}$ equal zero if at least one index is zero):

$$\chi_G^{00}(0)=-\frac{1}{2}\sqrt{-g}g_{\alpha\beta,kt}g^{00}E^{\alpha\beta kt}+\frac{1}{4}\sqrt{-g}g_{\mu\nu,k}g_{\alpha\beta,t}\left[g^{00}\left(-\frac{1}{2}E^{\mu\nu\alpha\beta}e^{kt}+E^{kt\alpha\nu}e^{\mu\beta}+2E^{\alpha\beta\nu t}e^{\mu k}\right)\right.$$

$$\left.-\frac{1}{2}E^{\mu\nu\alpha\beta}g^{k0}g^{t0}+E^{kt\mu\beta}g^{\alpha0}g^{\nu0}+E^{\mu tk\alpha}g^{\beta0}g^{\nu0}+2E^{\alpha\beta k\mu}g^{\nu0}g^{t0}\right]. \quad (36)$$

According to Dirac, this $L_G(0)$ is the only source of contributions to the scalar constraint and he constructed it using the simplifying assumption of (32) and later concluded that it "must hold also when g_{0r} does not vanish". Let us check this assertion by explicitly separating all space and time indices in (36)

$$\chi_G^{00}(0)=-\frac{1}{2}\sqrt{-g}g_{mn,kt}g^{00}E^{mnkt}+\frac{1}{4}\sqrt{-g}g_{mn,k}g_{pq,t}g^{00}\left(-\frac{1}{2}E^{mnpq}e^{kt}+E^{ktpn}e^{mq}+2E^{pqnt}e^{mk}\right)$$

$$+\frac{1}{4}\sqrt{-g}g_{m0,k}g_{0q,t}g^{00}g^{00}\left(E^{ktmq}+E^{mtkq}\right)+\frac{1}{2}\sqrt{-g}g_{m0,k}g_{pq,t}g^{00}\left[E^{pqkm}g^{t0}+g^{p0}\left(E^{ktmq}+E^{mtkq}\right)\right]$$

$$+\frac{1}{2}\sqrt{-g}g_{mn,k}g_{pq,t}\left(-\frac{1}{2}E^{mnpq}g^{k0}g^{t0}+E^{ktmq}g^{p0}g^{n0}+E^{mtkp}g^{q0}g^{n0}+2E^{pqkm}g^{n0}g^{t0}\right). \quad (37)$$

Some terms in (37) have explicit dependence on the space-time components of the metric tensor and these components will disappear only if condition (32) is imposed. For $\chi_G^{0k}(0)$ there are even more such components. Even with condition (32), the result is not zero and this part of the Hamiltonian, $L_G(0)$, contributes to the vector constraint.

Now let us find contributions coming from the second part, $L_X(0)$. After a rearrangement of the terms given in (24) into a form which is more suitable for calculation, we obtain

$$L_X(0)=\frac{1}{2}\frac{\sqrt{-g}}{g^{00}}\left[\frac{1}{2}E^{rsab}g_{rs,u}g_{ab,v}g^{u0}g^{v0}-g_{a\beta,b}g_{rs,u}\left(E^{rsab}+E^{rsba}\right)g^{u0}g^{\beta0}\right.$$

$$+g_{r\alpha,s}g_{a\beta,b}\left(E^{rsab}+E^{rsba}\right)g^{\beta 0}g^{\alpha 0}\right].\tag{38}$$

For the $\frac{\delta L_X(0)}{\delta g_{0\sigma}}$ part we calculate

$$\chi_X^{0\sigma}(0)=\left\{p^{0\sigma},-L_X(0)\right\}=$$

$$\frac{1}{2}\sqrt{-g}\delta_a^\sigma\left(-g_{rs,ub}E^{rsab}g^{u0}+g_{r\alpha,sb}E^{rsab}g^{\alpha 0}\right)+C^\sigma(eee)+C_I^\sigma(ee)+C_{II}^\sigma(ee).\tag{39}$$

The variation $\frac{\delta L_X(0)}{\delta g_{0\sigma}}$ obviously produces contributions which are only third and second order in $e^{\alpha\beta}$ as in (33). For terms of third order we find

$$C^\sigma(eee)=-\frac{1}{4}\sqrt{-g}g_{\mu\nu,k}g_{\alpha\beta,t}$$

$$\times\left[g^{0k}\left(e^{\beta\mu}E^{\sigma\nu t\alpha}+e^{\sigma\nu}E^{\alpha t\beta\mu}\right)+g^{0\alpha}\left(e^{\mu\nu}E^{\sigma\beta tk}+2e^{\nu\beta}E^{\sigma t\mu k}-2e^{\nu t}E^{\sigma\beta\mu k}\right)\right]\tag{40}$$

and in second order we have two contributions: the first proportional to $g^{0\sigma}$

$$C_I^\sigma(ee)=\frac{1}{4}\sqrt{-g}\frac{g^{0\sigma}}{g^{00}}\left[\frac{1}{2}g_{rs,u}g_{ab,v}E^{rsab}g^{u0}g^{v0}-g_{a\beta,b}g^{\beta 0}\left(E^{rsab}+E^{rsba}\right)\left(g_{rs,u}g^{u0}-g_{r\alpha,s}g^{\alpha 0}\right)\right]\tag{41}$$

and the second with an index σ on $E^{rs\sigma k}$

$$C_{II}^\sigma(ee)=\frac{1}{4}\sqrt{-g}g_{\mu\nu,k}\frac{g^{0\nu}g^{0\mu}}{g^{00}}\left[\frac{1}{2}g_{rs,t}\left(E^{rs\sigma k}+E^{rsk\sigma}\right)g^{t0}-g_{r\beta,t}g^{\beta 0}\left(E^{\sigma krt}+E^{\sigma ktr}\right)\right].\tag{42}$$

Note, that we cannot present the part quadratic in $e^{\alpha\beta}$ (41, 42) in a compact form, where terms with derivatives are a common factor, because of the mixture of four and three indices, which is the result of the original noncovariant modification (8) of the Lagrangian. When performing these calculations we have to consider all possible combinations separately.

It is not difficult to confirm that $\chi_X^{0\sigma}(0)$ is not zero, even with assumption (32), there are contributions to both the constraints $\chi_X^{00}(0)$ and $\chi_X^{0k}(0)$. Consequently, Dirac's conjecture, if made separately for $L_X(0)$ and $L_G(0)$, is not correct; but, when both parts are combined, the contribution of zeroth order to the secondary constraint is greatly simplified

$$\chi^{0\sigma}(0) = \chi_G^{0\sigma}(0) + \chi_X^{0\sigma}(0) =$$

$$\frac{1}{2}\sqrt{-g}g^{0\sigma}\left[-g_{mn,kt}E^{mnkt} + \frac{1}{4}g_{mn,k}g_{pq,t}\left(-E^{mnpq}e^{kt} + 2E^{ktpn}e^{mq} + 4E^{pqnt}e^{mk}\right)\right]. \qquad (43)$$

The $\chi^{00}(0)$-part is the same with or without condition (32) and $\chi^{0k}(0)$ is given by (43) with $\sigma = k$ (it is zero when (32) is imposed). Frequently (43) is written in a different form which is based on the following observation: if in the expression for the four-dimensional Ricci scalar R

$$R = g^{\alpha\beta}g^{\mu\nu}R_{\alpha\mu\beta\nu} = g_{\alpha\beta,\mu\nu}\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu}\right) - \frac{1}{4}g_{\alpha\beta,\gamma}g_{\mu\nu,\rho}$$

$$\times\left(g^{\alpha\beta}g^{\mu\nu}g^{\gamma\rho} - 3g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho} + 2g^{\alpha\rho}g^{\beta\nu}g^{\gamma\mu} + 4g^{\alpha\gamma}g^{\mu\rho}g^{\beta\nu} - 4g^{\alpha\gamma}g^{\beta\rho}g^{\mu\nu}\right), \qquad (44)$$

we keep only the spatial indices and change the covariant component of g^{km} to e^{km} or, equivalently, impose the conditions (32), we obtain the expression shown in square brackets of (43), which is often called $R_{(3)}$.

Equation (43) gives contributions to the secondary constraints of zeroth order in the momenta p^{km}. There are obviously contributions to χ^{0k}. Dirac's vector constraint, $\mathcal{H}^r$, does not have such contributions, so it is not directly related to the time development of the corresponding primary constraint p^{0k} (we will discuss this later).

For $\chi^{00}(0)$, the equation (43) has to be compared to the corresponding expression of Dirac's (D39):

$$X_L(0) = -B + \left\{\sqrt{-g}{g^{00}}^{1/2}g_{rs,u}E^{rusv}\right\}_{,v} \qquad (45)$$

where B (D38)

$$B = \frac{1}{4}\sqrt{-g}{g^{00}}^{1/2}g_{rs,u}g_{ab,v}\left\{E^{rasb}e^{uv} + 2E^{ruab}e^{sv}\right\} \qquad (46)$$

is a part of full expression (5) where after passing to "$e-$form" only the terms cubic in $e^{\alpha\beta}$ are present. Terms quadratic and linear in $e^{\alpha\beta}$ are neglected, which results from the simplifying assumption because all non-cubic terms have either g^{0k} or the derivatives $g_{0k,m}$. In his final

expressions (45) Dirac keeps e^{km}, not g^{km}, which is consistent with his statement that this has to be true without the simplifying assumption which removes the difference between e^{kn} and g^{kn}. In addition, we keep $g = \det(g_{\mu\nu})$ in all equations. Dirac used $J^2 = -\det(g_{\mu\nu})$ and $K^2 = -\det(g_{km})$ (or, probably, now more familiar notation ${}^4g$ for $\det(g_{\mu\nu})$ and g (or $g_{(3)}$) for $\det(g_{km})$) which are connected by $g^{00}J^2 = K^2$ or $\sqrt{-g} = \sqrt{-\det(g_{km})/g^{00}}$.

By differentiating the second term of (45), it is not difficult to derive the relation

$$\chi^{00}(0) = \frac{1}{2} g^{00^{1/2}} X_L(0). \tag{47}$$

Dirac's scalar constraint $\mathcal{H}_L$ ($\mathcal{H}_L(0) = X_L(0)$) is not the result of a direct calculation of $\{p^{00}, H_G\}$. This difference is not important for the proof of closure of the Dirac procedure and one can always consider linear combinations of constraints. For Castellani's procedure (or any other procedure) for finding gauge transformations we have to be careful with such redefinitions as we will demonstrate in the next Section.

Until now, we have been concerned with the most complicated contributions to the secondary constraints which are zeroth order in the momenta. Let us now consider the contributions to all orders. In the other two orders we obtain (using (23))

$$\chi^{0\sigma}(2) = \frac{\delta}{\delta g_{0\sigma}} H_G(2) = \frac{1}{2}\frac{1}{\sqrt{-g}}\frac{g^{0\sigma}}{g^{00}}\left(g_{ra}g_{sb} - \frac{1}{2}g_{rs}g_{ab}\right)p^{rs}p^{ab}, \tag{48}$$

$$\chi^{0\sigma}(1) = \frac{\delta}{\delta g_{0\sigma}} H_G(1) = -\delta_u^\sigma\left(p^{us}_{,s} - \frac{1}{2}e^{uv}p^{rs}g_{rs,v} + e^{uv}p^{rs}g_{rv,s}\right). \tag{49}$$

Note, as $\chi^{00}(1) = 0$ there are no contributions linear in the momenta to the scalar constraint, but $\chi^{0k}(2) \neq 0$, unless we impose (32).

$\chi^{0\sigma}(0)$ was already calculated in (43). For the full scalar constraint, χ^{00}, the relation (47) is preserved in all orders

$$\chi^{00} = \frac{1}{2} g^{00^{1/2}} \mathcal{H}_L. \tag{50}$$

The vector constraint χ^{0k} has non-zero contributions in all orders of the momenta unless (32) is imposed. Before we continue to compare our direct calculations with those of Dirac, let us try to present the canonical Hamiltonian as a linear combination of the secondary constraints we calculated above.

We approach this problem by considering different orders in the momenta. The highest order is the second and the result is easily obtained from the first terms of (23) and (48)

$$H_G(2) = \frac{1}{g^{00}\sqrt{-g}}\left(g_{ra}g_{sb} - \frac{1}{2}g_{rs}g_{ab}\right)p^{rs}p^{ab} = 2g_{0\sigma}\chi^{0\sigma}(2) \tag{51}$$

(using $g_{0\sigma}g^{0\sigma} = \delta_0^0 = 1$).

By considering (27), which is equivalent to the second terms of (23) and (49), we have in first order

$$H_G(1) = -2g_{u0}\left(p^{us}_{,s} - \frac{1}{2}e^{uv}p^{rs}g_{rs,v} + e^{uv}p^{rs}g_{rv,s}\right) = 2g_{0\sigma}\chi^{0\sigma}(1). \tag{52}$$

$H_G(2)$ and $H_G(1)$ are of the same form and we anticipate $H_G(0)$ is also in this form. Unfortunately, this is not obvious and we have to perform some calculations to show it. To preserve the structure found in (51, 52), we will demonstrate that

$$H_G(0) = -L_G(0) - L_X(0) = 2g_{0\sigma}\chi^{0\sigma}(0) + (...)_{,k}. \tag{53}$$

Note, that for $H_G(1)$ given by (27) we also obtain (52) only up to a total spatial derivative. Using (25), (38), and (43) we have

$$-L_G(0) - L_X(0) - 2g_{0\sigma}\chi^{0\sigma}(0) =$$

$$\left[\sqrt{-g}E^{mnki}g_{mn,i} - \sqrt{-g}g_{\mu\nu,i}\left(e^{\nu k}\frac{g^{0i}g^{0\mu}}{g^{00}} - e^{\nu i}\frac{g^{0k}g^{0\mu}}{g^{00}}\right)\right]_{,k}. \tag{54}$$

This equation demonstrates that the relations found for $H_G(2)$ and $H_G(1)$ are also valid for $H_G(0)$ and the canonical Hamiltonian can be written in terms of $\chi^{0\sigma}$as

$$H_G = 2g_{0\sigma}\chi^{0\sigma}. \tag{55}$$

Of course, this is correct up to total temporal (see (8)) and spatial (see (8), (27), and (54)) derivatives. The modification of the initial Lagrangian (8) was proposed by Dirac while (54) is obtained in the course of preserving relations found among contributions of higher order in the momenta to the constraints and the Hamiltonian. It would be very difficult to guess (54) without knowing the final result. Such an additional integration appearing in (54), is very often performed at the Lagrangian level. For example, in the book by Gitman and

Tyutin [17], in addition to Dirac's (8) (which are B and first term of C^i of Eq. (4.4.12) in [17]), the integrations of (54) were performed at the Lagrangian level (the second and third terms of C^i). The integrations of (54) can be derived only in the course of the Hamiltonian procedure, but such integrations (if they are known) are also correct when applied to the Lagrangian because (going back to Dirac's derivation) it is clear that $L_X(0)$ was constructed before the elimination of the velocities (i.e., at the Lagrangian level).

How is this covariant form of H_G (55) (which is equivalent to what was found in [30]) related to Dirac's expression for the Hamiltonian? Are they equivalent? The relationship between scalar constraints χ^{00} and $\mathcal{H}_L$ was found in (50); we now consider the relation between the vector constraints.

Let us inspect the form of our constraints calculated to different orders appearing in (48), (49), and (43). There are simple relations between the contributions of different orders to χ^{00} and χ^{0k}:

$$\chi^{0k}(2) = \frac{g^{0k}}{g^{00}}\chi^{00}(2), \quad \chi^{00}(1) = 0, \quad \chi^{0k}(1) \equiv \psi^{0k}, \quad \chi^{0k}(0) = \frac{g^{0k}}{g^{00}}\chi^{00}(0)$$

that allow one to write (to all orders)

$$\chi^{0k} = \psi^{0k} + \frac{g^{0k}}{g^{00}}\chi^{00} \tag{56}$$

with

$$\psi^{0k} = -p^{ks}_{,s} - e^{kv}p^{rs}\left(\frac{1}{2}g_{rs,v} - g_{rv,s}\right). \tag{57}$$

Solving (56) for ψ^{0k} gives a combination of the constraints χ^{00} and χ^{0k} which were originally calculated from the time development of the corresponding primary constraints.

In terms of this combination of constraints ψ^{0k} and χ^{00}, we obtain a different form of the canonical Hamiltonian

$$H_G = 2\frac{1}{g^{00}}\chi^{00} + 2g_{0k}\psi^{0k}. \tag{58}$$

This form of H_G is easy to compare with Dirac's, because his vector constraint is simply related to ψ^{0k}

$$2\psi^{0k} = e^{ks}\mathcal{H}_s. \tag{59}$$

For χ^{0k} we find

$$\chi^{0k} = \frac{1}{2}e^{ks}\mathcal{H}_s - \frac{1}{2}g_{0s}e^{sk}g^{00^{1/2}}\mathcal{H}_L. \tag{60}$$

Equation (59), together with (47), demonstrates the equivalence of the two different forms of H_G given in (55) and (58) to Dirac's canonical Hamiltonian

$$H_G = 2g_{0\sigma}\chi^{0\sigma} = 2\frac{1}{g^{00}}\chi^{00} + 2g_{0k}\psi^{0k} = \left(g^{00}\right)^{-1/2}\mathcal{H}_L + g_{r0}e^{rs}\mathcal{H}_s. \tag{61}$$

We would like to emphasize that Dirac's constraints are not a direct result of the time development of the primary constraints $\phi^{0\sigma}$ which produce $\chi^{0\sigma}$((50) and (60)). The only place known to us where this is stated is in the book by Gitman and Tyutin (Eq. (4.4.19) of [17]); but Dirac's particular combinations of constraints and the corresponding form of the Hamiltonian are usually used.

The linear approximation of $\chi^{0\sigma}$ gives exactly the constraints of linearized GR [37]. In the linearized case there is no difference between χ^{0k} and ψ^{0k}; therefore linearized gravity can provide little "guidance" to full GR, in contrast to what was emphasized by ADM in [41]. Any such guidance has to be taken cautiously.

To demonstrate closure of the Dirac procedure, any form of the canonical Hamiltonian (61) is suitable as they are all equivalent; and any linear combination of constraints can be used for this purpose (e.g., $\chi^{00} = \frac{1}{2}g^{00^{1/2}}\mathcal{H}_L$ and $\psi^{0k} = \frac{1}{2}e^{rs}\mathcal{H}_s$). When using Castellani's procedure to derive the gauge transformations generated by first-class constraints, we have to consider those secondary constraints that directly follow from the corresponding primary ones and the PBs of secondary constraints with the total Hamiltonian, not just with its canonical part (this is also discussed in the next Section).

All of Dirac's secondary constraints have a zero PB with the primary constraints. In constraint dynamics this means that Lagrange multipliers cannot be found at this stage. As the PB of the secondary constraints with the canonical part of the Hamiltonian is zero or proportional to constraints, the procedure is closed. This is exactly the case here when we are taking into account the algebra[7] of PBs among Dirac's combinations of the secondary constraints:

[7] This algebra is called "hypersurface deformation algebra" or "Dirac algebra" and can be found in many places, e.g. [10, 19].

$$\left\{\mathcal{H}_L\left(x\right),\mathcal{H}_L\left(x'\right)\right\}=e^{rs}\left(x\right)\mathcal{H}_s\left(x\right)\delta_{,r(x)}\left(x-x'\right)-e^{rs}\left(x^{'}\right)\mathcal{H}_s\left(x'\right)\delta_{,r(x')}\left(x-x'\right),$$

$$\left\{\mathcal{H}_s\left(x\right),\mathcal{H}_L\left(x'\right)\right\}=\mathcal{H}_L\left(x\right)\delta_{,s(x)}\left(x-x'\right), \tag{62}$$

$$\left\{\mathcal{H}_r\left(x\right),\mathcal{H}_s\left(x'\right)\right\}=\mathcal{H}_s\left(x\right)\delta_{,r(x)}\left(x-x'\right)-\mathcal{H}_r\left(x'\right)\delta_{,s(x')}\left(x-x'\right).$$

When dealing with the "covariant" secondary constraints $\chi^{0\sigma}$, the multipliers are again not determined, but now we have

$$\left\{\chi^{0\sigma},p^{0\gamma}\right\}=\frac{1}{2}g^{\sigma\gamma}\chi^{00}. \tag{63}$$

The closure of Dirac's procedure is obviously preserved when using the covariant constraints because $\chi^{0\sigma}$ and Dirac's constraints are simply related by (50) and (60). This can also be shown by direct calculation of $\{\chi^{0\sigma},H_G\}$ without any reference to Dirac's combinations of constraints and their algebra. These calculations are long and to perform them we found it more convenient to work in the intermediate stages with χ^{00} and ψ^{0k}. This allows us to sort out terms uniquely, and at the final stage we can express the result, using (56), in terms of covariant constraints. The details of such calculations will be given in [42]. We arrive to the following PBs of $\chi^{0\sigma}$ with the canonical part of the Hamiltonian

$$\left\{\chi^{00},H\right\}=-\frac{2}{\sqrt{-g}}I_{kmrb}p^{km}g_{0a}e^{ab}\chi^{0r}+\chi^{0k}_{,k}+\frac{g^{0\alpha}g^{0\beta}}{g^{00}}g_{\alpha\beta,k}\chi^{0k}-\frac{1}{2}g^{0b}g_{00,b}\chi^{00} \tag{64}$$

and

$$\left\{\chi^{0k},H\right\}=\frac{1}{\sqrt{-g}}\frac{1}{g^{00}}\left(2g_{ra}p^{ak}\chi^{0r}-g_{ab}p^{ab}\chi^{0k}\right)-\frac{g^{0k}}{g^{00}}\frac{2}{\sqrt{-g}}I_{tmrb}p^{tm}g_{0a}e^{ab}\chi^{0r}$$

$$+g^{0k}g_{00,t}\chi^{0t}+2g_{0p,t}g^{pk}\chi^{0t}+\frac{g^{0p}}{g^{00}}g^{kq}\left(g_{pq,r}+g_{rp,q}-g_{rq,p}\right)\chi^{0r}-\frac{1}{2}g^{km}g_{00,m}\chi^{00}. \tag{65}$$

Of course, we can present (64) and (65) as one "covariant" equation

$$\left\{\chi^{0\sigma}, H\right\} = -\frac{g^{0\sigma}}{g^{00}}\frac{2}{\sqrt{-g}}I_{tmrb}p^{tm}g_{0a}e^{ab}\chi^{0r} + \delta_m^\sigma\frac{1}{\sqrt{-g}}\frac{1}{g^{00}}\left(2g_{ra}p^{am}\chi^{0r} - g_{ab}p^{ab}\chi^{0m}\right) + \delta_0^\sigma\chi^{0k}_{,k}$$

$$-\frac{1}{2}g^{\sigma b}g_{00,b}\chi^{00} + g^{0\sigma}g_{00,t}\chi^{0t} + 2g_{0p,t}g^{p\sigma}\chi^{0t} + \frac{g^{0p}}{g^{00}}g^{\sigma q}\left(g_{pq,r} + g_{rp,q} - g_{rq,p}\right)\chi^{0r}. \qquad (66)$$

These equations, (64, 65) or (66), along with (63), provide proof of the closure of the Dirac procedure: higher order (tertiary) constraints do not appear and multipliers cannot be found because

$$\left\{\chi^{0\sigma}, H_T\right\} \sim \chi^{0\sigma}.$$

The Dirac Hamiltonian for GR, which is based on the modified Lagrangian of (10) and the simplifying assumption (32), is equivalent to the result of direct calculations given in (55) which are performed without any reference to surfaces of constant time. All the equivalent forms of the Hamiltonian of (61) are *only* the consequence of an initial modification that does not affect the equations of motion and preserves the four-dimensional symmetry. It is natural to expect that Dirac's Hamiltonian formulation, which is obtained without any *a priori* assumptions and restrictions (e.g. surfaces of constant time), has to preserve another manifestation of four-dimensional symmetry: invariance under the diffeomorphism transformation (1). Such a demonstration, given in the next Section, is an important consistency check of our results. All constraints are first-class and thus to find the generators of the gauge transformation, we have to consider "chains" of constraints. This means that one has to work, not with some combination of the constraints $\chi^{0\sigma}$, but with the exact results for $\{\phi^{0\sigma}, H_T\}$ and $\{\chi^{0\sigma}, H_T\}$. These results are complicated, especially (66), but their correctness can be verified if they lead to diffeomorphism invariance. A simple preliminary check of (66) is that the linearized version of this equation gives

$$\left\{\chi^{00}, H\right\} = -p^{0k}_{,k}, \quad \left\{\chi^{0k}, H\right\} = 0.$$

This is equivalent to the results of [37] (note, that in the linearized case $\chi^{0k}_{lin} = \psi^{0k}_{lin}$) and it leads to a linearized version of diffeomorphism invariance.

To summarize, the reality of Dirac's formulation, that is based on modifications of the initial Lagrangian, which do not affect the equations of motion, is as follows: notwithstanding

Dirac's references to space-like surfaces, all of his calculations were performed without use of any such surfaces. Consequently, Hawking's statement about the contradiction of the Hamiltonian formulation, based on splitting space-time into three spatial dimensions and one time dimension, is not applicable to Dirac's Hamiltonian formulation of GR, which does preserve the spirit of GR. Our own criticism of Dirac's formulation in [43] was not correct as we based it only on the 'interpretational' aspects of his work. This *faux pas* is also an illustration of how interpretations or some geometrical (or any other) reasonings can be dangerous if the "rule of procedure" referred by Lagrange is neglected.

Dirac's simplifying assumption, $g_{0k} = 0$ and (32), for constructing zeroth order in momenta contributions to the secondary constraints is not correct with respect to the individual parts given in (33, 39); but remarkably when these parts are combined together in (43), they are equivalent to his final expression. His secondary constraints do not follow directly from the time development of the primary constraints but rather they are particular combinations of the true secondary constraints $\chi^{0\sigma}$. His secondary constraints cannot be directly used to find gauge transformations (Dirac did not consider himself this question). In the next Section we will show that the generator built from the true constraints gives the four-dimensional diffeomorphism (1), and we can say that the true constraints of the Dirac formulation and their algebra *is* "the algebra of four diffeomorphisms" [29].

III. THE GAUGE GENERATOR AND TRANSFORMATION OF THE METRIC TENSOR

The knowledge of the complete set of first-class constraints (primary, $p^{0\sigma}(9)$, and secondary, $\chi^{0\sigma} = \chi^{0\sigma}(2) + \chi^{0\sigma}(1) + \chi^{0\sigma}(0)$, where contributions of different order in momenta are given by (48), (49) and (43)), as well as the PBs between the primary and secondary constraints (63), and the exact form of the closure (66) are sufficient to find the generators of the gauge transformations. This possibility is Dirac's old conjecture [7] which became a well developed algorithm and exists in a few variations [10, 11, 12]. We follow the work of Castellani [10] where the first application of such a method to Yang-Mills theory and ADM gravity[8] was considered.

[8] In [10] the author referred to the Dirac formulation of GR but in fact considered the ADM formulation. The non-equivalence of these two formulations will be discussed in the next Section.

Castellani's procedure is based on a derivation of the generator of gauge transformations which is defined by *chains* of first-class constraints. One starts with primary first-class constraint(s), $i = 1, 2, ...$, and construct the chain(s) $\xi_i^{(n)} G_{(n)}^i$ where $\xi_i^{(n)}$ is the nth order time derivative of the gauge parameter ξ_i ($n = 0, 1, ...$). The maximum value of n corresponds to the length of the chain (e.g., $n = 0, 1, 2$ for the system with tertiary constraints). The number of gauge parameters ξ_i is equal to the number of first-class primary constraints. Note, that these chains are an unambiguous construction once the primary constraints are defined; the remaining members of the chain are uniquely determined.

From this point, we specialize to the Dirac Hamiltonian formulation of GR with $n = 0, 1$ and $i = 0, 1, ..., (d-1)$; if $d = 4$ there are four primary and four secondary constraints.[9] The functions $G_{(n)}^i$ are calculated as follows

$$G_{(1)}^{\sigma}(x) = p^{0\sigma}(x), \tag{67}$$

$$G_{(0)}^{\sigma}(x) = +\left\{p^{0\sigma}(x), H_T\right\} + \int \alpha_{\gamma}^{\sigma}(x, y)\, p^{0\gamma}(y)\, d^3y \tag{68}$$

where the functions $\alpha_{\gamma}^{\sigma}(x, y)$ have to be chosen in such a way that the chain beginning with $G_{(1)}^{\sigma}$ in (67) ends on the primary constraint surface

$$\left\{G_{(0)}^{\sigma}, H_T\right\} = primary. \tag{69}$$

The generator $G(\xi_\sigma)$ is given by

$$G(\xi_\sigma) = \xi_\sigma G_{(0)}^{\sigma} + \xi_{\sigma,0} G_{(1)}^{\sigma}. \tag{70}$$

There are some peculiarities that arise when applying this algorithm to GR that cannot be seen in simpler cases like Maxwell, Yang-Mills or linearized GR theories.

Firstly, we comment on the use of different linear combinations of constraints. The term $\{p^{0\sigma}(x), H_T\}$ in (68) is uniquely defined by the choice of primary constraints. After Dirac's modification (8) of the original Lagrangian, these remain just the momenta $p^{0\sigma}$ conjugate to the $g_{0\sigma}$ components of the metric tensor. Direct calculation of the PBs of these primary constraints with the Hamiltonian gives

[9] The following calculations, as well as the results of the previous Section, are valid in all dimensions, except $d = 2$.

$$\left\{p^{00}\left(x\right),H_G\right\}=\chi^{00}=\frac{1}{2}g^{00^{1/2}}\mathcal{H}_L,$$

$$\left\{p^{0k}\left(x\right),H_G\right\}=\chi^{0k}=\psi^{0k}+\frac{g^{0k}}{g^{00}}\chi^{00}=\frac{1}{2}e^{ks}\mathcal{H}_s+\frac{1}{2}\left(g^{00}\right)^{-1/2}g^{0k}\mathcal{H}_L,$$

and these expressions, χ^{00} and χ^{0k}, must be used when the gauge generators are derived. Of course, one can use Dirac's combinations, $\mathcal{H}_L$ and $\mathcal{H}_s$, but only with appropriate coefficients or in appropriate combinations because of these inequalities: $\left\{p^{00},H_G\right\}\neq\mathcal{H}_L$, $\left\{p^{0k},H_G\right\}\neq e^{ks}\mathcal{H}_s$. We are not aware of any other situation where one must consider combinations of secondary constraints in ordinary field theories. Such a situation does not appear, for example, when the gauge generator for Yang-Mills is constructed [10]. In this case, the first term of (68) is just a secondary constraint which is the result of direct calculation of the PB of primary constraints with the Hamiltonian. In the Hamiltonian formulation of GR it is quite common (if not exclusive) to use of the Dirac combinations of constraints; but one has to be careful when gauge generators are constructed using $\mathcal{H}_L$ and $\mathcal{H}_s$.

Secondly, in both the Maxwell and Yang-Mills theories it is possible to choose the functions $\alpha\left(x,y\right)$ so that chains truly end with zero in (69) [10]. This is not the case for GR, and chains end only on the surface of the primary constraints. The effect of such a difference will be seen in our calculation of the gauge generators and the associated transformations.

Thirdly, the *total* Hamiltonian should be used in Castellani's procedure, not just its canonical part (see (68, 69)). Again, in linearized GR, Yang-Mills and Maxwell theories this difference is irrelevant because in these theories the PBs of secondary constraints with primary ones are zero. This is not the case for full GR as can be seen from (63) and similarly from equation (16) of [30].

Finally, a purely technical comment. There is a change of sign in front of the first term of (68) relative to that used in [10]. This is the result of Dirac's convention for the fundamental brackets in (31) (it is the negative of the fundamental brackets used in [10]).

To construct the generator (70) we have to find functions $\alpha_\gamma^\sigma\left(x,y\right)$ using the condition (69)

$$\left\{G_{(0)}^\sigma,H_T\right\}=\left\{\chi^{0\sigma}\left(x\right)+\int\alpha_\gamma^\sigma\left(x,y\right)p^{0\gamma}\left(y\right)d^3y,H_T\right\}=$$

$$\left\{\chi^{0\sigma}\left(x\right),H_T\right\}+\int\left\{\alpha^{\sigma}_{\gamma}\left(x,y\right),H_T\right\}p^{0\gamma}\left(y\right)d^3y+\int\alpha^{\sigma}_{\gamma}\left(x,y\right)\left\{p^{0\gamma}\left(y\right),H_T\right\}d^3y. \tag{71}$$

Part of the first term has already been calculated and $\left\{\chi^{0\sigma}\left(x\right),H_G\right\}$ is given by (66). For the part involving the primary constraints, $p^{0\gamma}$, we use (63) which gives

$$\left\{\chi^{0\sigma}\left(x\right),g_{00,0}p^{00}+2g_{0m,0}p^{0m}+H_G\right\}=\frac{1}{2}g_{00,0}g^{0\sigma}\chi^{00}+g_{0m,0}g^{\sigma m}\chi^{00}$$

$$-\frac{g^{0\sigma}}{g^{00}}\frac{2}{\sqrt{-g}}I_{tmrb}p^{tm}g_{0a}e^{ab}\chi^{0r}+\delta^{\sigma}_{m}\frac{1}{\sqrt{-g}}\frac{1}{g^{00}}\left(2g_{ra}p^{am}\chi^{0r}-g_{ab}p^{ab}\chi^{0m}\right)+\delta^{\sigma}_{0}\chi^{0k}_{,k}$$

$$-\frac{1}{2}g^{\sigma b}g_{00,b}\chi^{00}+g^{0\sigma}g_{00,t}\chi^{0t}+2g_{0p,t}g^{p\sigma}\chi^{0t}+\frac{g^{0p}}{g^{00}}g^{\sigma q}\left(g_{pq,r}+g_{rp,q}-g_{rq,p}\right)\chi^{0r}. \tag{72}$$

The second term of (71) is irrelevant because it is automatically zero on the surface of primary constraints, as required by (69).

For the last term of (71), taking into account the zero value of the PB among primary constraints, we find that

$$\left\{p^{0\gamma},H_T\right\}=\left\{p^{0\gamma},H_G\right\}=\chi^{0\gamma}. \tag{73}$$

This illustrates the advantage of using the 'covariant' constraints $\chi^{0\gamma}$ for deriving the generator. With (72-73), we can now read off the functions $\alpha^{\sigma}_{\gamma}\left(x,y\right)$ from (71) that compensates (72)

$$-\alpha^{\sigma}_{\gamma}\left(x,y\right)=\frac{1}{2}g_{00,0}\left(x\right)g^{0\sigma}\left(x\right)\delta^{0}_{\gamma}\left(x,y\right)+g_{0m,0}g^{\sigma m}\delta^{0}_{\gamma}$$

$$-\frac{g^{0\sigma}}{g^{00}}\frac{2}{\sqrt{-g}}I_{tmrb}p^{tm}g_{0a}e^{ab}\delta^{r}_{\gamma}+\delta^{\sigma}_{m}\frac{1}{\sqrt{-g}}\frac{1}{g^{00}}\left(2g_{ra}p^{am}\delta^{r}_{\gamma}-g_{ab}p^{ab}\delta^{m}_{\gamma}\right)+\delta^{\sigma}_{0}\delta^{k}_{\gamma,k}$$

$$-\frac{1}{2}g^{\sigma b}g_{00,b}\delta^{0}_{\gamma}+g^{0\sigma}g_{00,t}\delta^{t}_{\gamma}+2g_{0p,t}g^{p\sigma}\delta^{t}_{\gamma}+\frac{g^{0p}}{g^{00}}g^{\sigma q}\left(g_{pq,r}+g_{rp,q}-g_{rq,p}\right)\delta^{r}_{\gamma}, \tag{74}$$

where $\delta^{r}_{\gamma}\left(x,y\right)\equiv\delta^{r}_{\gamma}\delta\left(x-y\right)$ and the arguments x and y are explicitly written only in the first term. Contracting $\alpha^{\sigma}_{\gamma}\left(x,y\right)$ with the primary constraints and performing integration in the second term of (68) we obtain

$$G^{\sigma}_{(0)} = \chi^{0\sigma} - \frac{1}{2} g_{00,0} g^{0\sigma} p^{00} - g_{0m,0} g^{\sigma m} p^{00}$$

$$+ \frac{g^{0\sigma}}{g^{00}} \frac{2}{\sqrt{-g}} I_{tmrb} p^{tm} g_{0a} e^{ab} p^{0r} - \delta^{\sigma}_{m} \frac{1}{\sqrt{-g}} \frac{1}{g^{00}} \left(2 g_{ra} p^{am} p^{0r} - g_{ab} p^{ab} p^{0m} \right) - \delta^{\sigma}_{0} p^{0k}_{,k}$$

$$+ \frac{1}{2} g^{\sigma b} g_{00,b} p^{00} - g^{0\sigma} g_{00,t} p^{0t} - 2 g_{0p,t} g^{p\sigma} p^{0t} - \frac{g^{0p}}{g^{00}} g^{\sigma q} \left(g_{pq,r} + g_{rp,q} - g_{rq,p} \right) p^{0r}. \quad (75)$$

Equation (75) completes the calculation of the generator (70). Now the transformation of fields can be found by calculating their PB with the generator[10]

$$\delta \left(field \right) = \left\{ field, G \right\}. \quad (76)$$

For the time-time component of the metric tensor, g_{00}, we obtain (using Dirac's convention of (31) and keeping only part of the generator (70) with terms proportional to p^{00})

$$\delta g_{00} = \left\{ g_{00}, G \right\} = - \frac{\delta}{\delta p^{00}} G =$$

$$- \frac{\delta}{\delta p^{00}} \left(- \xi_{\sigma} \left(g_{00,0} \frac{1}{2} g^{0\sigma} p^{00} + g_{0m,0} g^{\sigma m} p^{00} - \frac{1}{2} g^{\sigma b} g_{00,b} p^{00} \right) + \xi_{0,0} p^{00} \right) =$$

$$- \xi_{0,0} + \left(\frac{1}{2} g_{00,0} g^{00} + g_{0m,0} g^{0m} \right) \xi_{0} + \left(\frac{1}{2} g_{00,0} g^{k0} + g_{0m,0} g^{km} \right) \xi_{k} - \frac{1}{2} g^{0b} g_{00,b} \xi_{0} - \frac{1}{2} g^{km} g_{00,m} \xi_{k}. \quad (77)$$

Let us compare this result with diffeomorphism invariance (1) which can be written in an equivalent form, that is more convenient for comparison with our calculations:

$$\delta_{(diff)} g_{\mu\nu} = - \xi_{\mu,\nu} - \xi_{\nu,\mu} + g^{\alpha\beta} \left(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta} \right) \xi_{\alpha}. \quad (78)$$

Taking $\mu = \nu = 0$ and explicitly separating space and time indices, we have

[10] Some authors defined transformations as $\delta \left(field \right) = \left\{ G, field \right\}$ which seems to be more natural. However, we keep the convention of Castellani that, of course, affects only an overall sign in the final result, which can always be incorporated into the gauge parameters (this is not a field dependent redefinition, as is used in [19]).

$$\delta_{(diff)}g_{00} = -2\xi_{0,0} + \left(g^{00}g_{00,0} + 2g^{0k}g_{0k,0} - g^{0k}g_{00,k}\right)\xi_0 + \left(g^{k0}g_{00,0} + 2g^{km}g_{0m,0} - g^{km}g_{00,m}\right)\xi_k. \tag{79}$$

We see that (79) is equivalent to (77) up to a numerical factor 2,

$$2\left\{g_{00}, G\right\} = \delta_{(diff)}g_{00}, \tag{80}$$

that can be incorporated into the gauge parameter by a rescaling $\xi_\sigma \to 2\xi_\sigma$.

Similarly, for the space-time components, g_{0k}, we have

$$\delta g_{0k} = \{g_{0k}, G\} =$$

$$-\frac{\delta}{\delta p^{0k}}\left[-\xi_{0\sigma}\left(-\frac{g^{0\sigma}}{g^{00}}\frac{2}{\sqrt{-g}}I_{tmrb}p^{tm}g_{0a}e^{ab}p^{0r} + \delta^\sigma_m\frac{1}{\sqrt{-g}}\frac{1}{g^{00}}\left(2g_{ra}p^{am}p^{0r} - g_{ab}p^{ab}p^{0m}\right) + \delta^\sigma_0 p^{0m}_{,m}\right.\right.$$

$$\left.\left.+g^{0\sigma}g_{00,t}p^{0t} + 2g_{0p,t}g^{p\sigma}p^{0t} + \frac{g^{0p}}{g^{00}}g^{\sigma q}\left(g_{pq,r} + g_{rp,q} - g_{rq,p}\right)p^{0r}\right) + \xi_{m,0}p^{0m}\right] =$$

$$= -\frac{1}{2}\xi_{0,k} - \frac{1}{2}\xi_{k,0}$$

$$+\frac{1}{2}\xi_{0\sigma}\left(-\frac{g^{0\sigma}}{g^{00}}\frac{2}{\sqrt{-g}}I_{tmkb}p^{tm}g_{0a}e^{ab} + \frac{1}{\sqrt{-g}}\frac{1}{g^{00}}\left(2g_{ka}p^{am}\delta^\sigma_m - g_{ab}p^{ab}\delta^\sigma_k\right)\right.$$

$$\left.+g^{0\sigma}g_{00,k} + 2g_{0p,k}g^{p\sigma} + \frac{g^{0p}}{g^{00}}g^{\sigma q}\left(g_{pq,k} + g_{kp,q} - g_{kq,p}\right)\right). \tag{81}$$

There is a difference between the transformation (81) and the transformation of the time-time component (77) as the momenta p^{ab} are present in (81). Using the definition of momenta (18) and re-expressing e^{km} in terms of g^{km} by (15), we obtain

$$\delta g_{0k} = -\frac{1}{2}\xi_{0,k} - \frac{1}{2}\xi_{k,0}$$

$$+\frac{1}{2}\left[g^{00}g_{00,k} + g^{0m}\left(g_{0m,k} + g_{km,0} - g_{0k,m}\right)\right]\xi_0$$

$$+\frac{1}{2}\left[g^{m0}\left(g_{00,k}+g_{k0,0}-g_{0k,0}\right)+g^{mn}\left(g_{0n,k}+g_{kn,0}-g_{0k,n}\right)\right]\xi_m, \tag{82}$$

which again equals $\delta_{(diff)}g_{\mu\nu}$ as given in (78) with $\mu\nu = 0k$; which is true provided we again rescale ξ_σ by a factor of two.

The last transformation to be checked is the transformation of the space-space components

$$\delta g_{km} = \{g_{km}, G\} = -\frac{\delta}{\delta p^{km}} G\left(p^{pq}\right). \tag{83}$$

The relevant part of the generator (70) which has an explicit dependence on p^{pq} is

$$G\left(p^{pg}\right) = \xi_\sigma\left[\chi^{0\sigma} + \frac{g^{0\sigma}}{g^{00}}\frac{2}{\sqrt{-g}}I_{tmrb}p^{tm}g_{0a}e^{ab}p^{0r} - \delta_m^\sigma\frac{1}{\sqrt{-g}}\frac{1}{g^{00}}\left(2g_{ra}p^{am}p^{0r} - g_{ab}p^{ab}p^{0m}\right)\right] \tag{84}$$

where parts of the secondary constraints ($\chi^{00}(2)$, $\chi^{0k}(2)$ and $\chi^{0k}(1)$) will also contribute to the final result. The variation of the last two terms in (84) gives contributions proportional to the primary constraints (which equal zero on the surface of primary constraints). The only relevant parts of the generator for δg_{km} are given by

$$G = \xi_0\chi^{00}(2) + \xi_k\left(\chi^{0k}(2) + \chi^{0k}(1)\right). \tag{85}$$

Performing variation of (85) with respect to p^{km}, using the expression for momenta given in (18), and reverting from e^{km} to g^{km} using (15), we obtain

$$\delta g_{km} = -\frac{1}{2}\left(\xi_{k,m} + \xi_{m,k}\right)$$

$$+\frac{1}{2}g^{00}\left(g_{k0,m} + g_{m0,k} - g_{km,0}\right)\xi_0 + \frac{1}{2}g^{p0}\left(g_{k0,m} + g_{m0,k} - g_{km,0}\right)\xi_p$$

$$+\frac{1}{2}g^{0p}\left(g_{kp,m} + g_{mp,k} - g_{km,p}\right)\xi_0 + \frac{1}{2}g^{pq}\left(g_{kq,m} + g_{mq,k} - g_{km,q}\right)\xi_p. \tag{86}$$

It is not difficult to check that up to the same numerical factor 2 (as occurred in (80)) this is equivalent to $\delta_{(diff)}g_{\mu\nu}$ (78) with $\mu\nu = km$.

We see that transformations of the time-time and space-time components of the metric tensor are exactly equivalent to a diffeomorphism and the space-space components give a

diffeomorphism only on the surface of primary constraints. Such a deviation from ordinary field theories like Yang-Mills can be expected because the derivation of generators is performed (i.e., the functions $\alpha_{\gamma}^{\sigma}(x, y)$ are found) only on a surface of primary constraints. This is a consequence of the peculiarities of diffeomorphism transformations that will be discussed at the end of this Section.

Returning to Dirac's statement [5] about abandoning four–dimensional symmetry in his approach; we can see that it is restrictive and only related to his initial modification of the Lagrangian (8). This abandoning of four-dimensional symmetry does not happen, neither in linearized [37] nor in full GR [30]. The Dirac Hamiltonian formulation of GR, as we demonstrated in this Section, allows one to derive the transformation of the metric tensor in covariant form and four-dimensional symmetry is preserved. The exact meaning of common statements, such as the one that is found in [29], "unfortunately, the canonical treatment breaks the symmetry between space and time in general relativity", must also be clarified in light of our results. Of course, and this is a property of the Hamiltonian approach itself, the four-dimensional symmetry is not broken, it just is not manifest. For any generally covariant theory with first-class constraints we can only make a conclusion about abandoning such a symmetry if the gauge invariance that is derived from the first-class constraints cannot be presented in covariant form. This will be shown to happen in the case of the ADM formulation. If the symmetry presented in the original Lagrangian disappears in the Hamiltonian formulation, then it should be considered as a very strong indication that there is a mistake in the formulation; it is not a problem with the initial Lagrangian or with the Hamilton-Dirac method. From this point of view, if the "canonical treatment breaks the symmetry", then such a treatment is not canonical.

Let us return to the derivation of the gauge transformations. Our derivation of the transformation was based on an application of Castellani's method [10]. There are at least two variations of it: one of them is based on the extended Hamiltonian [11, 18], where all first-class constraints are included, and the other, [12], is based only on the total Hamiltonian (i.e., only primary first-class constraints are included) as in Castellani's case. The equivalence of the algorithms [11] and [12] was discussed in [12] and a comparison of methods [11] and [10] was made in [11]. Primary constraints play a special role in all of these methods. The need to include multipliers associated with the primary constraints was also emphasized in [11, 18] and their importance in gauge transformations was demonstrated by some simple examples.

In [12], the multipliers are also important elements of this method (see below). Recently, the method of [12] was applied to the ADM Hamiltonian in [19], where the transformation of the metric tensor was derived and was shown to differ in form from (1). For completeness, we apply this method to the Dirac Hamiltonian formulation of GR to demonstrate that the derivation of the diffeomorphism transformation in the Hamiltonian approach is not an artifact of a particular procedure for finding a gauge generator. At the same time this demonstration will illustrate the equivalence between the two different methods described in [10] and [12], as well as the equivalence both of them to the Lagrangian treatment of this problem in [26].

The total Hamiltonian is the starting point of the method outlined in [12]

$$H_T = H_c + \lambda_\mu \phi^\mu. \tag{87}$$

For a system with only irreducible secondary first-class constraints and no tertiary constraints (so that $\{\phi^\mu, H_c\} = \chi^\mu$) the generator of gauge transformations is simply

$$G = \eta_\mu \phi^\mu + \xi_\mu \chi^\mu \tag{88}$$

with two sets of parameters, η_μ and ξ_μ (twice the number of primary constraints), which are related by (see Eq. (17) of [12]):

$$0 = \xi_{\mu,0} - \xi_\nu \left(V^\nu_{(s)\mu} + \lambda_\gamma B^{\nu\gamma}_{(s)\mu} \right) - \eta_\nu \left(W^\nu_{(s)\mu} + \lambda_\gamma C^{\nu\gamma}_{(s)\mu} \right). \tag{89}$$

Here W^ν_μ, V^ν_μ, $C^{\nu\gamma}_\mu$, and $B^{\nu\gamma}_\mu$ are the structure functions of the involutive algebra (see Eqs. (2) and (3) of [12])

$$\{H_c, \phi^\nu\} = W^\nu_{(p)\mu} \phi^\mu + W^\nu_{(s)\mu} \chi^\mu, \tag{90}$$

$$\{H_c, \chi^\nu\} = V^\nu_{(p)\mu} \phi^\mu + V^\nu_{(s)\mu} \chi^\mu, \tag{91}$$

$$\{\phi^\nu, \phi^\gamma\} = C^{\nu\gamma}_{(p)\mu} \phi^\mu + C^{\nu\gamma}_{(s)\mu} \chi^\mu, \tag{92}$$

$$\{\phi^\nu, \chi^\gamma\} = B^{\nu\gamma}_{(p)\mu} \phi^\mu + B^{\nu\gamma}_{(s)\mu} \chi^\mu. \tag{93}$$

The indices (p) and (s) indicate structure functions associated with the primary ϕ^{μ} and secondary constraints χ^{μ}, respectively. Note, that (89) involves only functions with the subscript (s), i.e. structure functions related to the primary constraints are not present, so that this equation is valid on the primary constraint surface. This is similar to what happens in Castellani's procedure.

To find the generator (88), one has to solve (89) for η_{ν}; and as in Castellani's method the number of independent parameters becomes equal to the number of primary constraints. For the Dirac Hamiltonian of GR, which is obtained after modification of the gamma-gamma part (with no effect on the equations of motion, canonical variables $g_{\mu\nu}$, and conjugate momenta $p^{\mu\nu}$) we have the simple primary constraints (9)

$$\phi^{0\mu} = p^{0\mu}$$

and the secondary constraints (for explicit expressions see (48), (49) and (43))

$$\left\{\phi^{0\mu}, H_c\right\} = \chi^{0\mu}. \tag{94}$$

This allows us to write the Hamiltonian in a compact and symmetric form

$$H_c = 2g_{0\mu}\chi^{0\mu}.$$

The possibility of solving (89) for η_{ν}, which is an ordinary algebraic equation, depends on the structure functions $W^{\nu}_{(s)\mu}$ and $C^{\nu\gamma}_{(s)\mu}$. In the case of the Dirac Hamiltonian formulation of GR (see (94)) they are

$$W^{\nu}_{(s)\mu} = \delta^{\nu}_{\mu}, \quad C^{\nu\gamma}_{(s)\mu} = 0 \tag{95}$$

and we can solve (89) for η_{μ}:

$$\eta_{\mu} = \xi_{\mu,0} - \xi_{\nu}\left(V^{\nu}_{(s)\mu} + \lambda_{\gamma}B^{\nu\gamma}_{(s)\mu}\right). \tag{96}$$

The structure functions $V^{\nu}_{(s)\mu}$ for GR are complicated (see (66)) and for $B^{\nu\gamma}_{(s)\mu}$ we have (see (63))

$$B^{\nu\gamma}_{(s)\mu} = \frac{1}{2}g^{\nu\gamma}\delta^{0}_{\mu},$$

and (96) becomes

$$\eta_{\mu}=\xi_{\mu,0}-\xi_{\nu}V_{(s)\mu}^{\nu}-\xi_{\nu}g_{0\gamma,0}\frac{1}{2}g^{\nu\gamma}\delta_{\mu}^{0}. \tag{97}$$

Note that the structure function $B_{(s)\mu}^{\nu\gamma}$ does not equal zero and the Lagrange multipliers, which are the velocities $g_{0\sigma,0}$ that cannot be expressed in term of $p^{0\sigma}$, enter (97) explicitly ($\lambda_0 = g_{00,0}$ and $\lambda_k = 2g_{0k,0}$).[11] After the substitution of (97) into (88) we obtain a one-parameter ξ_ν (the number of components equals to the number of primary constraints) generator

$$G(\xi_\nu)=-\xi_{\nu}g_{00,0}\frac{1}{2}g^{\nu 0}\phi^{0}-\xi_{\nu}g_{0m,0}g^{\nu m}\phi^{0}+\xi_{\nu,0}\phi^{\nu}-\xi_{\nu}V_{(s)\mu}^{\nu}\phi^{\mu}+\xi_{\nu}\chi^{\nu}. \tag{98}$$

This expression has to be compared to the generator found using Castellani's procedure

$$G=\xi_{\nu,0}G_{(1)}^{\nu}+\xi_{\nu}G_{(0)}^{\nu}. \tag{99}$$

The third term of (98) is exactly the same as the first term of (99); the first, second, and last terms of (98) are also the same as in Castellani's approach (see the first line of (75)). The fourth term of (98), to be compared with our result obtained using Castellani's approach, is $-\xi_\nu V_{(s)\mu}^{\nu}\phi^{\mu}$. The structure function, $V_{(s)\mu}^{\nu}$, originates from the calculation of the PB of (91), which in the case of a field theory is

$$\{H_c,\chi^{\nu}(x)\}=\int V_{(s)\mu}^{\nu}(x,y)\,\chi^{\mu}(y)\,d^3y. \tag{100}$$

The direct calculation of $\{H_c,\chi^\nu\}$ gives terms proportional to χ^μ and its derivatives; for its explicit form see (66)

$$\{H_c,\chi^{\nu}\}=K_{\mu}^{\nu}\chi^{\mu}+M_{\mu}^{\nu k}\chi_{,k}^{\mu}. \tag{101}$$

This equation is usually presented in the following form in order to find the structure functions

$$\{H_c,\chi^{\nu}\}=\int\left(K_{\mu}^{\nu}(x)\,\delta(x-y)-M_{\mu}^{\nu k}(x)\frac{\partial}{\partial y_k}\delta(x-y)\right)\chi^{\mu}(y)\,d^3y. \tag{102}$$

[11] Expressions for the multipliers came from the Legendre transformation and from the fact that the modified Lagrangian is independent of $g_{0\mu.0}$, so that $H=g_{\nu\mu,0}p^{\nu\mu}-L=g_{00,0}p^{00}+2g_{0\mu,0}p^{0\mu}+...$

This is the standard form for an intermediate result in such calculations (e.g. see [10]). To find the generator (98), it is not necessary to rewrite (101) in the form of (102); and the direct substitution $\chi^{\mu} \to \phi^{\mu}$ into (101) gives the corresponding part, $-\xi_{\nu} V^{\nu}_{(s)\mu} \phi^{\mu}$, of the generator (98). This is equivalent to the second and third lines of (75).

If the novel method of [12] is applied to the Dirac Hamiltonian of GR and not to the ADM Hamiltonian, as was done in [19], then the generator (98) is equivalent to the one obtained using the old procedure of [10]; and so they generate the same gauge transformation. We have found that the results for the gauge transformation are equivalent, whatever method is used to find it, when applied to the same Hamiltonian.

The peculiarities of Castellani's procedure, when it is applied to the Hamiltonian of GR, were discussed at the beginning of this Section and illustrated by a derivation of (1) using the methods [10] and [12]. They are originated from the algebra of constraints either in Dirac's formulation, given in this Section, or in the 'covariant' formulation of [30]. This algebra is different from that of ordinary gauge theories. It reflects the peculiarities of diffeomorphism invariance if compared to the gauge invariance of ordinary gauge theories. We now briefly consider this topic.

In the Introduction, we restrict our discussion to a particular meaning of diffeomorphism given in (1) that is generally accepted in literature on GR and which is similar to the usual gauge transformations. It is in exactly this sense that diffeomorphism invariance can be derived from the Hamiltonian approach. Now we would like to describe this transformation without recourse to the Hamiltonian formulation as it is usually presented in textbooks (e.g. see [20]); and we wish to reveal how a difference between these two views of diffeomorphism invariance manifests itself. This exercise will also demonstrate the connection between (1) and general coordinate transformations.

The principle of general covariance, the cornerstone of GR, puts severe restrictions on the possible forms of the Lagrangian. The simplest is the EH Lagrangian.[12] The EH action and the Einstein equations are invariant under a general coordinate transformation

$$x'^{\mu} = f^{\mu}\left(x^{\nu}\right) \tag{103}$$

and the corresponding transformation of the metric tensor

[12] Of course, it is not unique and there are many posibilities: such as Lovelock gravity [44] or $f\left(R\right)$.

$$g'^{\mu\nu}\left(x'\right)=\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}g^{\alpha\beta}\left(x\right). \tag{104}$$

For infinitesimal transformations

$$x^{\mu}\rightarrow x'^{\mu}=x^{\mu}+\xi^{\mu}\left(x\right) \tag{105}$$

(104) can be written as

$$g'^{\mu\nu}\left(x'\right)=g^{\mu\nu}\left(x\right)+\xi^{\nu}_{,\alpha}g^{\mu\alpha}\left(x\right)+\xi^{\mu}_{,\alpha}g^{\alpha\nu}\left(x\right)+O\left(\xi^{2}\right). \tag{106}$$

Note that the components ξ^{μ}, form a true vector [38, 45], in contrast to the parameters $\varepsilon^{\perp}$ and ε^{k} which appear in the ADM formulation of [24] (see (2) and its derivation in the next Section).

If we consider $\xi^{\mu}\left(x\right)$ as being a small parameter and restrict our interest to the first-order contributions in ξ^{μ}, then the exact invariance of the EH action and the Einstein equations of motion is lost as only the inclusion of all terms in the expansion of (106) will preserve it. In addition, if we want to present (106) in a form similar to a gauge transformation, in which the invariance with respect to replacement of variables is in the same coordinate frame of reference [46], we should write both sides of (106) in the same coordinate system. This can be done by an additional approximation, using the Taylor expansion of $g'_{\mu\nu}\left(x'\right)$:

$$g'^{\mu\nu}\left(x'\right)=g'^{\mu\nu}\left(x^{\alpha}+\xi^{\alpha}\left(x\right)\right)=g'^{\mu\nu}\left(x\right)+g^{\mu\nu}_{,\alpha}\xi^{\alpha}+O\left(\xi^{2}\right) \tag{107}$$

where in the second term (which is already linear in ξ^{α}) we used $\left.\frac{\partial g'^{\mu\nu}(x')}{\partial x'^{\alpha}}\right|_{x'=x}=g^{\mu\nu}_{,\alpha}+O\left(\xi\right)$.

Combining (106) and (107) and keeping only terms linear in ξ^{α}, we obtain

$$\delta g^{\mu\nu}=g'^{\mu\nu}\left(x\right)-g^{\mu\nu}\left(x\right)=\xi^{\nu}_{,\alpha}g^{\mu\alpha}\left(x\right)+\xi^{\mu}_{,\alpha}g^{\alpha\nu}\left(x\right)-g^{\mu\nu}_{,\alpha}\xi^{\alpha} \tag{108}$$

which is equivalent to

$$\delta g^{\mu\nu}=\xi^{\mu;\nu}+\xi^{\mu;\nu}. \tag{109}$$

By repeating similar calculations, or by using $\delta\left(g_{\mu\alpha}g^{\alpha\nu}\right)=0$, one can find transformations for the covariant components of a metric tensor

$$\delta g_{\mu\nu} = -\xi_{\mu;\nu} - \xi_{\nu;\mu} = -\xi_{\mu,\nu} - \xi_{\nu,\mu} + 2\Gamma^{\alpha}_{\mu\nu}\xi_{\alpha}. \tag{110}$$

This equation is just (1) (or its equivalent form (78) that was more convenient for comparison with our calculations). With ξ^{μ}, being a true vector, the transformations (109) and (110) are generally covariant (these are covariant derivatives of a true vector), so they are independent of the choice of coordinate system; these are the transformations we derived from the Dirac Hamiltonian of GR and in [30].

Similarly, we can obtain the transformation of the Christoffel symbols. Using the relation between $\Gamma^{\alpha}_{\mu\nu}$ and $g_{\mu\nu}$ in (13) and the transformation $\delta g_{\mu\nu}$ in (110), we obtain

$$\delta\Gamma^{\alpha}_{\mu\nu} = -\xi^{\beta}\Gamma^{\alpha}_{\mu\nu,\beta} + \Gamma^{\beta}_{\mu\nu}\xi^{\alpha}_{,\beta} - \Gamma^{\alpha}_{\mu\beta}\xi^{\beta}_{,\nu} - \Gamma^{\alpha}_{\nu\beta}\xi^{\beta}_{,\mu} - \xi^{\alpha}_{,\mu\nu}. \tag{111}$$

Note that the presence of second-order derivatives of the parameters ($\xi^{\alpha}_{,\mu\nu} = \partial_{\mu}\partial_{\nu}\xi^{\alpha}$) immediately allows one to come to a general conclusion about the constraint structure of the Hamiltonian formulation of GR in the first-order form, the Einstein affine-metric formulation of [3], where $g^{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$ are treated as independent fields.[13] The presence of the second-order derivatives of the parameters in the transformation $\delta\Gamma^{\alpha}_{\mu\nu}$ implies that the generators must have the same order of derivatives, i.e. the tertiary constraints must appear in such a formulation. Of course, direct calculations confirmed this simple observation [43, 51, 52, 53]. In the first discussion of the Hamiltonian formulation of the first-order form of GR which was presented in [41], the tertiary constraints did not appear because some first-class constraints were solved *before* closure of Dirac's procedure was reached; this procedure is not a consistent implementation of Dirac's procedure for treating constrained systems. This fact was clearly demonstrated in the Appendix of [54] and was discussed in [43]. The loss of tertiary constraints is also due to a misleading analogy between the first-order formulations of Electrodynamics and linearized GR appearing in [41, 55]. In the first-order formulation of Electrodynamics, where the field strength is treated as an independent variable, there is no increase in the order of the derivatives of the gauge parameters in the generator of gauge transformations because this is relates to the fact that the variation of the field strength is zero. In contrast, the variation of $\Gamma^{\alpha}_{\mu\nu}$ under diffeomorphism transformations, (111), is not

[13] This formulation was originally introduced by Einstein [3] (for English translation see [47]), but mistakenly attributed to Palatini [48] (see also Palatini's original paper [49] and its English translation [50]).

zero and cannot even be written in covariant form as $\delta g_{\mu\nu}$ was in (110). This characteristic is consistent with $\Gamma^{\alpha}_{\mu\nu}$ not being a true tensor [20, 56].[14] Also there is no linear combination of the first-order derivatives of the metric tensor that is exactly invariant under diffeomorphism transformations, as well as under general coordinate transformations.[15] This invariance is also related to the fact that a generally covariant action for GR cannot be built from terms only quadratic in the first-order derivatives of the metric tensor; and the simplest generally covariant, EH Lagrangian, is proportional to a Ricci scalar and this involves second-order derivatives of the metric tensor. This is why the affine-metric formulation of GR leads unavoidably to tertiary constraints and consequently increases the length of the chain of constraints and the order of the derivatives in the gauge generator.

Let us compare (111) with the gauge invariance of Yang-Mills theory. We have the field strength $F^{a}_{\mu\nu}$ whose variation does not vanish, in contrast to Electrodynamics,

$$\delta F^{a}_{\mu\nu} = f^{abc}\theta^{b}F^{c}_{\mu\nu},$$

with f^{abc} a totally antisymmetric structure constant and θ^{b} a gauge parameter. We do not have exact invariance for $F^{a}_{\mu\nu}$, but the gauge parameters enter only linearly. Thus in the first-order formulation of Yang-Mills theory, if we consider the field strength as an independent variable, there is no increase in the length of the chains of constraints needed to accommodate these transformations. In GR it is not possible to build any combination of first-order derivatives which is exactly invariant under diffeomorphism; and it is also impossible to find a combination whose variation is proportional to the gauge parameter or its first-order derivatives.

The transformation (1) (or other equivalent forms) is written in the same coordinate system and, because this combination is a true tensor, it is independent of the coordinate transformations. In this sense it is "analogous to the gauge transformation" [57, 58]; but the absence of combinations of derivatives that do not lead to an increase of the order of the derivatives of gauge parameters in the generator is a distinct property of GR.

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[14] The $\Gamma^{\alpha}_{\mu\nu}$ behaves like a tensor only with respect to linear coordinate trasformations [20]. Probably, this was the origin of the analogy between Electrodynamics and linearized GR and of the conjecture that this analogy should be extended to full GR [41].

[15] Actually, such a combination exists which is a true tensor: this is the covariant derivative of the metric tensor, $g_{\mu\nu;\gamma}$, but it identically equals zero.

In addition, the Lagrangian of GR is not exactly invariant under a diffeomorphism transformation, in contrast to Maxwell and Yang-Mills theories. From (108) and (111) and by using

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu}$$

we can find the transformations of $R_{\mu\nu}$ and $R = g^{\alpha\beta}R_{\alpha\beta}$:

$$\delta R_{\mu\nu} = -\xi^{\rho} R_{\mu\nu,\rho} - \xi^{\rho}_{,\mu} R_{\nu\rho} - \xi^{\rho}_{,\nu} R_{\mu\rho}, \quad \delta R = -\xi^{\rho} R_{,\rho}. \tag{112}$$

From the above relations it is not difficult to obtain the transformation of the EH Lagrangian under a diffeomorphism

$$\delta\left(\sqrt{-g}R\right) = \left(-\xi^{\mu}\sqrt{-g}R\right)_{,\mu}. \tag{113}$$

This lack of exact invariance is distinct from what occurs in the Maxwell and Yang-Mills theories; but other models exist with Lagrangians which are also invariant up to a total divergence, e.g. Topologically Massive Electrodynamics (TME) of [59]. (See [60, 61, 62] for a discussion of its first-order formulation.) Despite differences in the invariance property of Lagrangians, which can be either exact as in ordinary Electrodynamics or up to a total divergence as in TME [59], the equations of motion are *exactly* invariant under gauge transformations. In GR the transformation of the equations of motion is proportional to the equations themselves [46]. Using (112) we can easily find transformations of the Einstein field equations

$$\delta G_{\mu\nu} = -\xi^{\rho} G_{\mu\nu,\rho} - \xi^{\rho}_{,\mu} G_{\nu\rho} - \xi^{\rho}_{,\nu} G_{\mu\rho}, \tag{114}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor. So, in GR under diffeomorphism transformations (1), the equations of motion are invariant only 'on-shell' which does not contradict the principle of gauge invariance: a solution to the equations of motion maps into a solution. This last result, the 'on-shell' invariance of the equations of motion might cause some confusion because (1) was obtained from infinitesimal coordinate transformations (105), which are a particular case of general coordinate transformations (103). The EH action and the Einstein equations are exactly invariant under (103), therefore, under (105) (in fact, the Einstein equations were originally postulated so as to satisfy (103)). After writing (105) in

the same coordinate system and using the approximations (106) and (107), exact invariance is lost. Such a mapping, (114), by itself, is not peculiar to GR because a similar property is present in the Yang-Mills theory where the transformation of the equations of motion is

$$\delta\left(D_\mu F_a^{\mu\nu}\right) = f_{abc}\theta_b D_\mu F_c^{\mu\nu}.$$

As in GR, the Yang-Mills field equation is only invariant 'on-shell'; but this transformation is proportional to the gauge parameter, whereas the transformation of $\delta G_{\mu\nu}$ in (114) also contains derivatives of the gauge parameter. The peculiar algebra of constraints in GR is related to this increase in the order of the derivatives of the gauge parameters in the transformations of the equations of motion, as well as the impossibility of finding a combination of derivatives of the metric tensor which is either exactly invariant or whose variation does not require derivatives of the gauge parameters.

On one hand, the diffeomorphism transformation (1) is related to coordinate transformations and, as a consequence, to general covariance. On the other hand, the Hamiltonian formulation of GR is performed using the same "rule of procedure" as in ordinary gauge theories, allowing one to derive the same transformation (1). The resulting invariance makes this transformations distinct from ordinary gauge theories because of the presence of derivatives of the gauge parameters. Such distinct transformations should manifest themselves in all steps of the procedure and this is exactly what we have found. For example, the non-zero PB among primary and secondary constraints in (63) and the 'on-shell' of primary constraints result of (84) are properties that are not observed in ordinary gauge theories. These peculiarities are present in Dirac's formulation as considered in this article, as well as in [30]; and both of these formulations allow one to derive the diffeomorphism invariance (1). At least two of the above mentioned properties, which are related to (63) and (84), are absent in the ADM formulation and the transformations derived from the ADM Hamiltonian are different from a diffeomorphism. A comparison between the Dirac and ADM formulations will be made in the next Section.

IV. THE DIRAC HAMILTONIAN OF GR VERSUS THE ADM HAMILTONIAN OF GEOMETRODYNAMICS

In the previous Section we demonstrated that, following the "rule of procedure" as applied to Dirac's Hamiltonian of GR (61) (with the covariant metric $g_{\mu\nu}$ and corresponding conjugate momentum $p^{\mu\nu}$ as the fundamental canonical variables) the diffeomorphism invariance (1) can be derived using Castellani's procedure [10] or the method of [12]. The same result was recently obtained in the Hamiltonian formulation of GR [30] without using Dirac's modifications (8) and in the Lagrangian approach of [17] by Samanta [26]. This result is exactly what one would expect for the invariance of GR, as well as the equivalence of results in the Hamiltonian and Lagrangian approaches.

In addition, because Dirac's non-covariant modifications of the Lagrangian do not change the equations of motion, four-dimensional symmetry is preserved and it is reflected in the covariant form of the transformations (1). We have also demonstrated that Dirac's references to space-like surfaces are not part of his actual calculations. As a result, Hawking's statement [27] that "the split into three spatial dimensions and one time dimension seems to be contrary to the whole spirit of relativity" is not related to Dirac's formulation, where only *manifest* invariance (but not the invariance itself) is broken by explicitly considering different components of the metric tensor. We have seen that working with the original Einstein variable, the metric tensor, we have the Hamiltonian formulation of GR that is consistent with diffeomorphism symmetry, and the spirit of GR is "alive".

There exists a more popular Hamiltonian which is based on a different set of variables: the lapse and shift functions and the space-space components of the metric tensor. It is frequently, but mistakenly called the Dirac Hamiltonian (e.g. [10]) and even its variables are called "Dirac's lapse and shift" [22].[16] This formulation (with the lapse and shift functions) is due to Arnowitt, Deser and Misner (see [6] and references therein). The name "Dirac-ADM" is also not correct; and despite the apparent similarities between the Dirac and ADM Hamiltonians, they are different (see below). The appropriate and best known names for the ADM formulation are "ADM gravity" and geometrodynamics, as opposed to Einstein gravity. The gauge transformation derived from the ADM Hamiltonian by the methods of

[16] The names "lapse" and "shift" were introduced neither by ADM nor by Dirac and appeared only later, for example, in [14, 32].

[10, 12], using ADM variables and constraints, is not diffeomorphism invariance (1). This fact was recently demonstrated very clearly using the method of [12] in [19] and was discussed in our Introduction. The field-dependent redefinition of the parameters (2) for the ADM Hamiltonian differs from the exact result that was obtained in Section III from the Dirac formulation, where no field-dependent redefinition was used, and in [26] and [30] where the question of equivalence with diffeomorphism does not arise.

We feel that it is insufficient to say that the formulations of [4, 30] and that of Dirac [5], both of which use the metric tensor as the canonical variable, correctly describe the Hamiltonian of GR and one is obliged to work with the original Einstein variables. One should not try to recast GR into a description of the motion of surfaces; more precisely, one should not change variables to make such an interpretation plausible. It is necessary to understand why two such closely related Hamiltonians, these of Dirac and ADM, which are mistakenly said to be equivalent, produce different results. This is a general question related to the Hamiltonian formulation of singular Lagrangians. The understanding of the peculiarities of the Hamiltonian method for constrained systems can protect from repeating some mistakes that have been made when considering formulations of such theories.

The analysis of the ADM formulation is also interesting from another point of view, as it provides a very instructive example of what might be called an "interpretational" approach to physics. In the original work of ADM (e.g., [6]), an interpretation of the variables they introduced was proposed; and in later work, this interpretation became the cornerstone of that theory. Attempts were made to "derive" results starting from that interpretation, i.e. by elevating the interpretation to the level of first principles [32]. Such an approach exhibits lack of rigour and relies on some geometrical reasoning. The essence of this approach is perfectly expressed in the following quotation from [63]: "I capture as much of the classical theory as I can by *pictorial visualization*"[17] and "The reader is encouraged to follow the broad outlines and not worry about technical details". The "advantage" of such an approach is that it cannot be disproved; yet it prevents one from obtaining any reliable prediction or result. For example, if we accept Dirac's references to space-like surfaces as a part of his formalism, then Hawking's statement that introduction of a family of space-

[17] This becomes a well-known pedagogical approach in teaching conceptional (not mathematical) physics for non-science students.

like surfaces "seems to be contrary to the whole spirit of relativity" [27] forces us to reject this formulation as an inappropriate formulation of GR. And yet, as we have demonstrated, Dirac's formulation leads to a direct restoration of diffeomorphism invariance and, because of this, it is consistent with the spirit of GR and is the correct Hamiltonian formulation of GR. This demonstration shows how a purely interpretational consideration can lead to a wrong conclusion and that the interpretational approach without having to "worry about technical details" is meaningless. Dirac's derivation follows the "rule of procedure" and allows us to check any interpretation by explicit calculations. If something is constructed only using pure interpretation, then the final result cannot be checked by calculations and can only be analyzed by comparing its consistency with general principles. An "interpretation" cannot serve as a ground to disprove a result and in fact could be wrong, as in the case of Dirac's formulation in which he makes references to space-like surfaces that were not used in his calculations. A general understanding of the limitations of the interpretational approach probably provides an explanation of why Hawking's words [27], spoken almost thirty years ago on the occasion of the centenary of Einstein's birth, were not enough to cause people to immediately abandon the ADM formulation.

Yet more disturbing, is that this "interpretational" language has completely prevailed in the Hamiltonian formulation of GR. As an example, consider the correct work of Samanta [26] in which he used the Lagrangian formulation, and where there are no surfaces of constant time, space-like, slicing, etc., then it is abruptly altered when he refers to the Hamiltonian formulation of the same theory and states that "slicing is essential for Hamiltonian formulation". This assertion is obviously wrong as slicing is *not essential* and the Hamiltonian formulations of GR obtained without any reference to slicing gives a consistent result, as we have demonstrated in the previous Section and in [30].

The general trend in Physics and the main goal of many physicists is the unification of theories and methods on all possible levels; but even now, when we are a few years away from the 100th jubilee of the discovery of GR [64] there remains an inconsistency when discussing of different formulations (Lagrange and Hamilton-Dirac) of the same theory, Einstein's GR! This leads to erroneous observations, such as "It is worth noting that generalized Hamiltonian dynamics is not completely equivalent to Lagrangian formulation of the original theory. In the Hamiltonian formalism the constraints generate transformations of phase space variables, however, the group of these transformations *does not have to be equivalent*

to the group of gauge transformations of Lagrangian theory" [65].

We consider it important to understand and find explicitly where and why the ADM Hamiltonian formulation contradicts the spirit of GR, and why it cannot be associated with Einstein's theory (i.e. it is not the Hamiltonian formulation of GR, but rather the Hamiltonian formulation of distinct theory: "geometrodynamics"). One can talk about abandoning the spirit of GR if one is discussing a different theory built on different principles (such as [66]) but the ADM formulation has been given the appearance of having a formal basis on GR (in view of their articles and the presentation in many textbooks); indeed the summary of ADM's work in [6] has the title "Dynamics of General Relativity". The mathematical manifestation of the spirit of Einstein's GR is the general covariance of Einstein's equations of motion for the metric tensor. Einstein's GR is a field theory and the methods used in ordinary field theories, those of Lagrange and Hamilton, if applied correctly, must not destroy its spirit. This is exactly what we want to investigate: where was "a regular and uniform rule of procedure" broken in the ADM approach. In Hamiltonian language, we want to see where the canonical procedure was destroyed by passing from the Dirac Hamiltonian to the ADM Hamiltonian and why the ADM formulation with their variables is not equivalent to GR or, in other words, is not a canonical formulation of GR.

It does not seem possible to start from the Lagrangian of GR, where surfaces are not present, and then after introducing new variables have such surfaces. We will follow a path distinct from the interpretational approach and pay attention to technical details by using the "uniform rule of procedure" in analyzing the ADM formulation. In the previous Section we demonstrated that, when we are using the rule of procedure, surfaces do not appear in either Dirac's calculations or in [30]. We will not repeat the calculations of the previous Section or of [30], but instead we will analyze how the ADM formulation is related to that of Dirac.

Let us compare the two Hamiltonians of Dirac and ADM. Castellani himself considered the GR and Yang-Mills theories as an illustrative examples of his algorithm for finding gauge transformations. Referring to Dirac's book [7], Castellani [10] started with the statement "from the Hilbert action one *derives* the Hamiltonian"[18]

[18] Such a Hamiltonian can be "derived" without recourse to the EH Lagrangian. We refer the reader interested in "visualization" or in the geometrical reasoning behind a derivation or geometrical meaning of this equation to the numerous figures in [32]. We disregard such approaches as inadequate for any

$$H^{ADM} = N\mathcal{H}_{\perp}^{ADM} + N^i\mathcal{H}_i^{ADM} + N_{,0}\Pi + N_{,0}^i\Pi_i, \tag{115}$$

where Π and Π_i are momenta conjugate to N and N^i, respectively.

This Hamiltonian never appears in Dirac's book or in any of his articles. Dirac's *derivation* of the Hamiltonian of GR is in the article [5] that we discussed in previous Sections. It is different from (115) and given by (61) which is the canonical part of the Hamiltonian, and his primary constraints are the momenta $p^{0\mu}$ conjugate to $g_{0\mu}$ (9). Equation (115) is, in fact, the ADM result and we have indicated so by using the superscript 'ADM'. In order to compare the results of the Dirac and ADM formulations we use a slightly different convention for the Dirac Hamiltonian which appeared in a subsequent article of Dirac (see Eq. (7) of [67] for the canonical part of Hamiltonian)

$$H_D = \left(-g^{00}\right)^{-1/2}\mathcal{H}_L + g_{r0}e^{rs}\mathcal{H}_s + g_{00,0}p^{00} + 2g_{0k,0}p^{0k}. \tag{116}$$

In this convention, g_{00} is negative [67].

Let us compare the secondary constraints of the two formulations. $\mathcal{H}_i^{ADM}$, the "diffeomorphism" constraint, is given in many sources (e.g. see Eq. (3.14b) of [6]) as

$$\mathcal{H}_i^{ADM} = g_{ik}\mathcal{H}_{ADM}^k = -2g_{ik}\Pi_{|j}^{kj}$$

where Π^{kj} is a momentum conjugate to the spatial metric g_{kj}. The symbol "|" seems to indicate the covariant derivative with respect to g_{kj} [6]; but the definition of the particular covariant derivative used in ADM is non-standard and is not easily found (see Eq. (5.1) of [68] and Eq. (2.3b) of [69] which are consequently papers six and twelve in their series)

$$\Pi_{|j}^{kj} \equiv \Pi_{,j}^{kj} + \Pi^{lm}\Gamma_{lm}^k$$

which gives

$$\mathcal{H}_i^{ADM} = -2g_{ik}\Pi_{,j}^{kj} - 2\Pi^{kj}g_{ik,j} + \Pi^{kj}g_{jk,i}.$$

This is exactly the Dirac constraint (see Eq. (D41) of [5]) or our (29)). We note that the ADM definition of a covariant derivative of a contravariant second rank tensor mimics a

proofs.

covariant derivative of a contravariant vector (the first rank tensor) or a covariant derivative of the tensor density [38]. If we use the standard definition of a covariant derivative of the second rank tensor [20, 38, 56] we will obtain a different result. This result can only be presented as a standard covariant derivative (but with respect to the spatial metric g_{ik} only) if we write it as

$$\mathcal{H}_i^{Dirac} = \mathcal{H}_i^{ADM} = -2g_{is}\sqrt{\det g_{km}}\left(\frac{\Pi^{ls}}{\sqrt{\det g_{km}}}\right)_{;l}$$

(see Eq. (37) of [70] or Eq. (E.2.34) of [15]).

The scalar, "Hamiltonian", constraint $\mathcal{H}_\perp^{ADM}$ is given by Eq. (3.14b) of [6]

$$\mathcal{H}_\perp^{ADM} = -\sqrt{g}\left[{}^3R + g^{-1}\left(\frac{1}{2}\Pi^2 - \Pi^{ij}\Pi_{ij}\right)\right] =$$

$$-\sqrt{g}\;{}^3R + \frac{1}{\sqrt{g}}\left(g_{ik}g_{jm} - \frac{1}{2}g_{ij}g_{km}\right)\Pi^{ij}\Pi^{km}. \tag{117}$$

Using (50) (employing the convention of [67] where g^{00} is negative), from (48) we obtain the second term of (117). Also from (43), and taking into account (44), the first term of (117) follows. Thus, this constraint is also equivalent to Dirac's $\mathcal{H}_L$.

Dirac's combinations $\mathcal{H}_L$ and $\mathcal{H}_s$ of the true secondary constraints $\chi^{0\sigma}$ (given in (50, 60)) are *exactly* the same as the ADM secondary[19] constraints $\mathcal{H}_\perp^{ADM}$ and $\mathcal{H}_i^{ADM}$

$$\mathcal{H}_\perp^{ADM} = \mathcal{H}_L, \quad \mathcal{H}_i^{ADM} = \mathcal{H}_i. \tag{118}$$

The only difference between the first two terms of (115) and (116) is that the field-dependent coefficients in front of Dirac combinations of constraints are called new variables by ADM. These are the lapse and shift functions

$$N \equiv \left(-g^{00}\right)^{-1/2}, \tag{119}$$

$$N^i \equiv g_{j0}e^{ji} = -\frac{g^{0i}}{g^{00}}. \tag{120}$$

[19] Actually, in the ADM formulation they appear as primary [6, 72]. We will return to this later.

In addition, Dirac's e^{ji} (15) (the reciprocal of g_{ji}) is called the "three-dimensional" metric g^{ji} [6]. (In fact, g^{ji} is the space-space component of the full four-dimensional metric in four-dimensional space-time.) To distinguish e^{ji} from the space-space components of Einstein's four-dimensional contravariant metric tensor, the latter is defined to be ${}^4g^{ji}$ [6]. Dirac's notation is more transparent as it arises in his derivation of the Hamiltonian that was analyzed in Section II.[20]

The confused notion (in the literature) that these two formulations are equivalent, is understandable, especially in light of relation (118). Another reason is that in many presentations of the ADM Hamiltonian (115), such as in [29], the primary constraints are ignored by imposing the idea that the lapse and shift variables are merely the Lagrange multipliers for the constraints $\mathcal{H}_\perp$ and $\mathcal{H}_i$ and that they can be treated as 'primary' rather than 'secondary' constraints. Firstly, in such an approach even derivation of the gauge transformations of all components of the metric tensor becomes impossible as we are no longer dealing with the full phase space of the Hamiltonian (e.g., see the remark on p. 3288 of [24]). The methods of derivation of Castellani [10] and of [12] (which is used in [19]) employ the complete phase space and so all fields and their conjugate momenta must be included. Secondly, dropping primary constraints contradicts the methods of constraint dynamics: primary constraints are first-class and must not be solved as this destroys the gauge invariance present in the Lagrangian. Only second-class constraints can be solved and then some of the variables can be eliminated provided PBs are replaced by Dirac brackets. All this means that if we derive generators of gauge transformations, following any procedure using $\mathcal{H}_\perp^{ADM}$ and $\mathcal{H}_i^{ADM}$ as the first-class primary constraints, we will have generators independent of the momenta conjugate to the "multipliers", so that the gauge transformation of, for example N, would be zero

$$\delta N = \{N, G\} = 0. \tag{121}$$

This result means that $\delta g^{00} = 0$, which is not related to a diffeomorphism transformation at all. Without the primary constraints the time derivatives of the lapse function, for example, would vanish according to the Hamiltonian formulation,

[20] The ADM renaming was probably introduced to underline the geometrical interpretation of their variables.

$$\{N, H\} = 0.$$

Thus, N is constant in time, yet if we use the total Hamiltonian (116), we obtain

$$\{N, H_T\} = N_{,0}.$$

Finally, in the Hamiltonian approach, the primary constraints come from varying the Lagrangian with respect to velocities, and if we follow this rule, then $\mathcal{H}_{\perp}^{ADM}$ and $\mathcal{H}_{i}^{ADM}$ are not primary constraints. We will consequently work with the total Hamiltonian.

Let us continue to compare these two formulations. The relation between their primary constraints is as yet unclear and we shall return to this later. Form (119) and (120) the metric and its inverse are (e.g., see [10, 14])

$$g_{\mu\nu} = \begin{pmatrix} g_{ij}N^iN^j - N^2 & g_{ij}N^j \\ g_{ij}N^j & g_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^i}{N^2} \\ \frac{N^i}{N^2} & {}^3g^{ij} - \frac{N^iN^j}{N^2} \end{pmatrix}. \tag{122}$$

The fundamental PB of the canonical variables, the components of the covariant metric tensor and their corresponding conjugate momenta, for Dirac's Hamiltonian are [5]

$$\left\{p^{\alpha\beta}(x), g_{\mu\nu}(x')\right\} = \frac{1}{2}\left(\delta_\mu^\alpha\delta_\nu^\beta + \delta_\nu^\alpha\delta_\mu^\beta\right)\delta_3(x - x'), \tag{123}$$

whereas for the ADM approach the fundamental PBs are (e.g., see [10, 19, 71])

$$\left\{g_{ij}(x), \Pi^{kl}(x')\right\} = \frac{1}{2}\left(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k\right)\delta_3(x - x') = \Delta_{ij}^{kl}\delta_3(x - x'), \tag{124}$$

$$\left\{N^i(x), \Pi_j(x')\right\} = \delta_j^i\delta_3(x - x'), \tag{125}$$

$$\{N(x), \Pi(x')\} = \delta_3(x - x'). \tag{126}$$

Other PBs presumably equal zero (i.e. $\left\{N(x), \Pi^{kl}(x')\right\} = 0$, etc.) if these variables are to be canonical.

We now investigate why the Dirac and ADM approaches to the canonical treatment of GR lead to diffeomorphism transformations in the former case and to transformations that correspond to a diffeomorphism only after a non-covariant field-dependent redefinition of gauge parameters in the latter case.

In Section III we have demonstrated that the gauge transformations that follow from Dirac's Hamiltonian can be derived using both the methods of [10] and [12]. The method [12] was applied to ADM gravity in [19]. The result of [19] is not new, but it is probably the first complete consideration of how one can derive the gauge transformations from the constraint structure of the ADM formulation. The application of Castellani's method to the ADM Hamiltonian given in Appendix of [10] is opaque and incomplete as the relation between the diffeomorphism and the ADM parameters is not explicitly given and only the transformations of $g_{0\mu}$ are found. The calculations themselves were performed in an unnatural way - in order to find the transformations of the metric tensor, the ADM variables were expressed in terms of the metric in the generator. For completeness, we shall use Castellani's approach to find the gauge generator with the ADM Hamiltonian and compare it with the result of [19]. In addition, we will show some of the peculiarities in this calculation which are related to the somewhat confusing notation used by ADM.

According to Castellani's procedure, the generators of a gauge transformation can be constructed for the Hamiltonian using the so-called algebra of the Dirac constraints (62) (unnumbered equation preceding Eq. (29) of [10])[21]

$$\{\mathcal{H}_\perp, H\} = N_{,r}e^{rs}\mathcal{H}_s + \left(Ne^{rs}\mathcal{H}_s\right)_{,r} + \left(N^r\mathcal{H}_\perp\right)_{,r}, \tag{127}$$

$$\{\mathcal{H}_i, H\} = N_{,i}\mathcal{H}_\perp + N^j_{,i}\mathcal{H}_j + \left(N^j\mathcal{H}_i\right)_{,j}. \tag{128}$$

These lead to the generator (see Eq. (29) of [10])

$$G = -\int \left\{\left[\varepsilon^\perp\left(\mathcal{H}_\perp + N_{,i}e^{ij}\Pi_j + \left(N\Pi_i e^{ij}\right)_{,j} + \left(\Pi N^j\right)_{,j}\right) + \varepsilon^\perp_{,0}\Pi\right]\right.$$
$$\left. + \left[\varepsilon^i\left(\mathcal{H}_i + N^j_{,i}\Pi_j + \left(N^j\Pi_i\right)_{,j} + N_{,i}\Pi\right) + \varepsilon^i_{,0}\Pi_i\right]\right\} d^3x. \tag{129}$$

There are some differences between these equations and the corresponding ones of [10] as we use Dirac's notation, e^{rs}, which in ADM is called the three-dimensional metric. Actually, this renaming is sloppy and confusing because (see (122)) ${}^4g^{km} = {}^3g^{km} - \frac{N^kN^m}{N^2} = {}^3g^{km} + \frac{g^{0k}g^{0m}}{g^{00}}$ which, when solved for ${}^3g^{km}$, is equivalent to Dirac's e^{km} (15). We keep e^{rs} to avoid

[21] From this point we are considering the ADM formulation and do not use the superscript ADM.

the temptation to raise spatial indices with this tensor. We can do this for the spatial metric (as they are inverses) but we cannot do this for derivatives, as was done in [10] in the second term of (129) where $N^{,j}\Pi_j$ is correct *only* if we consider $N^{,j}$ as short for $N_{,i}e^{ij}$. This is because, according to the standard rules of raising indices in GR,

$$\partial^j = g^{j\mu}\partial_\mu = g^{ji}\partial_i + g^{j0}\partial_0 = e^{ji}\partial_i + \frac{g^{0i}g^{0j}}{g^{00}}\partial_i + g^{j0}\partial_0, \tag{130}$$

and

$$\partial^j = e^{ji}\partial_i \tag{131}$$

only if $g^{j0} = 0$ which is Dirac's simplifying assumption (32). If we use (131) instead of (130) we would obtain a different result.

The generator (129) allows one to find the transformations of the ADM fields (N, N^i and g_{km}) and then by using the definition of these variables (119, 120), we can *formally* find the transformations of $g_{\mu\nu}$. We do this in natural order - we first find the transformations of the ADM variables using the generator in terms of the ADM variables and then revert to the metric tensor. Transformations of the ADM fields are calculated using $\delta\left(field\right) = \{field, G\}$.

Starting with the simplest variable, N, we obtain

$$\delta_{ADM}N = \{N, G\} = \varepsilon^{\perp}_{,j}N^j - \varepsilon^{\perp}_{,0} - \varepsilon^i N_{,i} \tag{132}$$

which using $N = \left(-g^{00}\right)^{-1/2}$ gives

$$\delta_{ADM}N = \frac{1}{2}\left(-g^{00}\right)^{-3/2}\delta_{ADM}g^{00}$$

and so we find

$$\delta_{ADM}g^{00} = 2\left(-g^{00}\right)^{+3/2}\delta N = 2\left(-g^{00}\right)^{+3/2}\left[\varepsilon^{\perp}_{,j}\left(-g^{0j}/g^{00}\right) - \varepsilon^{\perp}_{,0} - \varepsilon^i\left(\left(-g^{00}\right)^{-1/2}\right)_{,i}\right]. \tag{133}$$

This differs from the diffeomorphism transformation (109):

$$\delta_{(diff)}g^{00} = 2\xi^{0,0} - \xi^0 g^{00}_{,0} - \xi^k g^{00}_{,k}.$$

In ADM variables we cannot restore diffeomorphism invariance; the most that can be done is to present (133) in a *form* similar to a diffeomorphism:

$$\delta_{ADM} g^{00} = 2\left(-g^{00}\right)^{+1/2}\left[g^{0j}\varepsilon^{\perp}_{,j} + g^{00}\varepsilon^{\perp}_{,0}\right] - \varepsilon^{i} g^{00}_{,i}$$

and using $\partial^0 = g^{0\mu}\partial_\mu = g^{00}\partial_0 + g^{0k}\partial_k$ this becomes

$$\delta_{ADM} g^{00} = 2\left(-g^{00}\right)^{+1/2}\varepsilon^{\perp,0} - \varepsilon^{i} g^{00}_{,i} =$$

$$2\left[\varepsilon^{\perp}\left(-g^{00}\right)^{+1/2}\right]^{,0} - \left[\varepsilon^{\perp}\left(-g^{00}\right)^{+1/2}\right] g^{00}_{,0} - \left[\varepsilon^{k} + \frac{g^{0k}}{g^{00}}\varepsilon^{\perp}\left(-g^{00}\right)^{+1/2}\right] g^{00}_{,k}.$$

The combinations in square brackets "correspond" to the diffeomorphism parameters

$$\xi^0 = \varepsilon^{\perp}\left(-g^{00}\right)^{+1/2} = \frac{1}{N}\varepsilon^{\perp}, \tag{134}$$

$$\xi^k = \varepsilon^k + \frac{g^{0i}}{g^{00}}\varepsilon^{\perp}\left(-g^{00}\right)^{+1/2} = \varepsilon^k - \frac{N^i}{N}\varepsilon^{\perp} \tag{135}$$

or

$$\varepsilon^{\perp} = N\xi^0 = \left(-g^{00}\right)^{-1/2}\xi^0, \tag{136}$$

$$\varepsilon^k = \xi^k + N^i\xi^0 = \xi^k - \frac{g^{0i}}{g^{00}}\xi^0. \tag{137}$$

Equations (134-137) are equivalent to the result of [19] (where the methods of [12] were used) and also to what is found in [21, 24]. The relations (134, 136) can be found in the Appendix of Castellani's article [10], but (135, 137) were not given there explicitly. This field-dependent redefinition of gauge parameters provides a "correspondence" [19], but not an equivalence with the diffeomorphism, that follows directly from consideration of the Dirac Hamiltonian.

For the next variable, N^k, we obtain

$$\delta_{ADM} N^k = \left\{N^k, G\right\} = -\left[\varepsilon^{\perp} N_{,j} e^{jk} - \varepsilon^{\perp}_{,j} N e^{kj} + \varepsilon^j N^k_{,j} - \varepsilon^k_{,j} N^j + \varepsilon^k_{,0}\right] \tag{138}$$

and using $N^k = -\frac{g^{0k}}{g^{00}}$ we find

$$\delta_{ADM} g^{0k} = \frac{g^{0k}}{g^{00}} \delta_{ADM} g^{00} + g^{00} \left[\varepsilon^{\perp} N_{,j} e^{jk} - \varepsilon^{\perp}_{,j} N e^{kj} + \varepsilon^{j} N^{k}_{,j} - \varepsilon^{k}_{,j} N^{j} + \varepsilon^{k}_{,0} \right].$$

After expressing N and N^k in terms of the metric

$$\delta_{ADM} g^{0k} = \frac{g^{0k}}{g^{00}} \delta_{ADM} g^{00}$$

$$+ g^{00} \left[\varepsilon^{\perp} \left[\left(-g^{00} \right)^{-1/2} \right]_{,j} e^{jk} - \varepsilon^{\perp}_{,j} \left(-g^{00} \right)^{-1/2} e^{kj} + \varepsilon^{j} \left(-\frac{g^{0k}}{g^{00}} \right)_{,j} - \varepsilon^{k}_{,j} \left(-\frac{g^{0j}}{g^{00}} \right) + \varepsilon^{k}_{,0} \right]$$

we again have an invariance that is not a diffeomorphism (109)

$$\delta_{(diff)} g^{0k} = \xi^{0,k} + \xi^{k,0} - \xi^{0} g^{0k}_{,0} - \xi^{m} g^{0k}_{,m}.$$

If we perform the field-dependent change of parameters (136, 137) we again can present $\delta_{ADM} g^{0k}$ in the *form* of a diffeomorphism transformation.

The transformation of the space-space components g_{km} was not considered in [10] because, according to the author, it is well-known that (129) generates a diffeomorphism transformation of g_{km}. Let us check this statement;

$$\delta g_{km} = \{ g_{km}, G \} = -\frac{\delta}{\delta \Pi^{km}} \left[\varepsilon^{\perp} \mathcal{H}_{\perp} + \varepsilon^{i} \mathcal{H}_{i} \right] \tag{139}$$

which, keeping only the Π^{pq}-dependent part of secondary constraints, gives

$$- \varepsilon^{\perp} \frac{1}{\sqrt{-g}} \left(-g^{00} \right)^{-1/2} \left(g_{ip} g_{jq} - \frac{1}{2} g_{ij} g_{pq} \right) 2 \Delta^{ij}_{km} \Pi^{pq} - 2 \left(\varepsilon^{i} g_{ip} \right)_{,q} \Delta^{pq}_{km} + \varepsilon^{i} \left(2 g_{pi,q} - g_{pq,i} \right) \Delta^{pq}_{km}. \tag{140}$$

We must express Π^{pq} in terms of g_{ij} and its derivatives; using Eq. (7-3.9b) of [6]

$$\Pi^{ij} = \sqrt{-{}^{4}g} \left({}^{4}\Gamma^{0}_{pq} - g_{pq} \; {}^{4}\Gamma^{0}_{rs} g^{rs} \right) g^{ip} g^{jq}$$

which is (taking into account that the "three-dimensional quantity" g^{ip} in ADM is Dirac's e^{ip})

$$\Pi^{ij} = \sqrt{-g} \left(\Gamma^{0}_{pq} e^{ip} e^{jq} - \Gamma^{0}_{rs} e^{rs} e^{ij} \right) = -\sqrt{-g} E^{rsab} \Gamma^{0}_{ab}. \tag{141}$$

This expression is equivalent to Dirac's expression for p^{ij} (18). Substituting (141) into (140) and using (14), (19) and (20), we obtain

$$\delta_{ADM} g_{km} = -\varepsilon^{\perp} \left(-g^{00}\right)^{-1/2} 2\Gamma^{0}_{km} - 2\left(\varepsilon^{i} g_{ip}\right)_{,q} \Delta^{pq}_{km} + \varepsilon^{i}\left(2g_{pi,q} - g_{pq,i}\right)\Delta^{pq}_{km},$$

or in the explicit form, using (18), for the ADM formulation

$$\delta_{ADM} g_{km} = \varepsilon^{\perp} \frac{1}{N}\left[N_{a,b} + N_{b,a} - g_{ab,0} - N^{k}\left(g_{ak,b} + g_{bk,a} - g_{ab,k}\right)\right]$$

$$- \varepsilon^{i}_{,m} g_{ik} - \varepsilon^{i}_{,k} g_{im} - \varepsilon^{i} g_{km,i} \tag{142}$$

and for Dirac's variables

$$\delta_{ADM} g_{km} = -\varepsilon^{\perp}\left(-g^{00}\right)^{-1/2}\left[g^{00}\left(g_{a0,b} + g_{b0,a} - g_{ab,0}\right) + g^{0k}\left(g_{ak,b} + g_{bk,a} - g_{ab,k}\right)\right]$$

$$- \varepsilon^{i}_{,m} g_{ik} - \varepsilon^{i}_{,k} g_{im} - \varepsilon^{i} g_{km,i}. \tag{143}$$

This is again different from the transformation of the spatial components of the metric under a diffeomorphism (110)

$$\delta_{(diff)} g_{km} = -g_{km,0}\xi^{0} - g_{k0}\xi^{0}_{,m} - g_{m0}\xi^{0}_{,k} - g_{km,i}\xi^{i} - g_{ki}\xi^{i}_{,m} - g_{mi}\xi^{i}_{,k}. \tag{144}$$

Again, only after the field-dependent redefinition of parameters (136, 137) we can obtain a "correspondence" between $\delta_{ADM} g_{km}$ and $\delta_{(diff)} g_{km}$. (Note that both parameters have to be redefined despite the apparent equivalence of the last three terms in both equations (143) and (144).) So, as in the case of Dirac's Hamiltonian, both methods [10, 12] produce the same result (2) for the ADM Hamiltonian.

We would like to note that even a spatial diffeomorphism does not follow directly from the ADM formulation despite what is often stated in the literature (e.g. [29]). If we treat the lapse and shift functions as 'multipliers'[22] and consider the secondary constraints as

[22] The authors of [14] in the "Historical remark" on p. 486 stated that "The great payoff of this work [ADM] was recognition of the lapse and shift functions of equation (21.40) [the same as in (21.42) or our (122)] as Lagrange multipliers, the coefficients of which gave directly and simply Dirac's constraints." As we

being primary (the contradictions that result from such manipulations have already been discussed), we do not correct this problem because a spatial diffeomorphism does not follow. In this case, according to Castellani's procedure, the generator is simply

$$G = \varepsilon^{\perp} H_{\perp} + \varepsilon^{i} H_{i} \tag{145}$$

which is equivalent to (139). Even in such a "formulation" the gauge parameters have to be redefined and, in addition, the transformations of lapse and shift functions equal zero (see, e.g. (121)).

There is only one way to "derive" spatial diffeomorphism invariance and it explains the origin of the term "diffeomorphism constraint". It behooves us to warn the reader that such a "derivation" has nothing to do with any procedure. If, in addition to eliminating primary constraints and promoting secondary to being primary (which leads to the generator (145)), we also consider only the second term of this generator, then

$$\delta g_{km} = \left\{ g_{km}, \varepsilon^{i} H_{i} \right\} \tag{146}$$

will give the spatial diffeomorphism (see the second line of (143)). The only possible explanation of why such manipulations were accepted is that it seems to follow the "guidance" which comes from linearized gravity. From the derivation of the gauge transformations for linearized gravity in [37], it is clear that the only part of the generator proportional to χ^{0n} contributes to the transformation of the space-space components of the metric tensor, but this is not the case for full GR.

Ironically, the result (146), which is just the consequence of a series of manipulations that contradict any consistent procedure, is often presented as being the "problem" of the Hamiltonian formulation of GR: "Hamiltonian" and "diffeomorphism" constraints are treated in a different manner [29, 84]. Furthermore, the conclusion is drawn, based on (146), that "the diffeomorphism constraint can be shown to be associated with the invariance of general relativity under spatial diffeomorphism" [29] (see also [71]). Finally, because (146) leads to

have shown, "the great payoff" of this recognition is that the ADM formulation lost the connection with GR and the gauge transformations derived from it are different from diffeomorphism. We would also like to mention that in the same "remark" the authors wrote: "Dirac paid no particular attention to any variational principle". The interested reader is encouraged to look at Dirac's papers, especially at [5], to recognize that this is not correct.

a spatial diffeomorphism, this result is interpreted as “disappearance of Diff$\mathcal{M}$” and considered as “the problem that has worried many people working in geometrodynamics for so long” [73].

Statements similar to the few of the forgoing quotations above can be found in many articles. They are based on questionable manipulations; but, at the same time, they clearly demonstrate that some authors correctly consider that the restoration of diffeomorphism invariance for all components of the metric tensor to be expected (however, “this expectation has never been fully realized...” [73])[23] and its absence is taken to be a deficiency or a contradiction arising in the Hamiltonian formulation. To have the possibility of restoring a transformation (whatever it may be) of all variables one must work in the full phase space and use all of the first-class constraints. We disregard “approaches” leading to (145) and (146) and return to the analysis of the total Hamiltonians (115) and (116).

We now ask why, in the ADM case, we must redefine the gauge parameters (134-137) to have a “correspondence” with diffeomorphism, whereas from the Dirac Hamiltonian (and also from the formulation without Dirac’s modifications [30]) diffeomorphism arises directly. Dirac’s Hamiltonian was obtained from the Lagrangian of GR, after some integrations that do not affect the equations of motion; and the ADM Hamiltonian apparently follows from the same Lagrangian. The two Hamiltonian formulations of the same Lagrangian ought to be equivalent and should give the same gauge transformation, which is not the case for the Dirac and ADM approaches.

It is well-known that different sets of phase space variables can be used to describe Hamiltonian systems. In the ordinary Classical Mechanics of non-singular systems (e.g., [74, 75]), for a given Hamiltonian $H\left(q_i, p_i\right)$ and the Hamilton equations,

$$q_i = \{q_i, H\} = \frac{\partial H}{\partial p_i},\ \ p_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}, \tag{147}$$

we can pass to another set of phase space variables (Q_i, P_i):

$$q_i = q_i\left(Q_k, P_k\right),\ \ p_i = p_i\left(Q_k, P_k\right),\ \ K\left(Q_k, P_k\right) = H\left(q_i\left(Q_k, P_k\right),\ p_i\left(Q_k, P_k\right)\right), \tag{148}$$

such that

[23] This expectation was fully realized in [30] and in the previous Section.

$$Q_i = \{Q_i, K\} = \frac{\partial K}{\partial P_i}, \;\; P_i = \{P_i, K\} = -\frac{\partial K}{\partial Q_i}. \tag{149}$$

If this system of Hamilton equations can be solved, then one can return to the original variables by using the inverse of (148)

$$Q_i = Q_i\left(q_k, p_k\right), \;\; P_i = P_i\left(q_k, p_k\right). \tag{150}$$

In Classical Mechanics, even for non-singular systems, it is well-known, that [74] "...at first sight we might think that arbitrary point transformations of the phase space are now at our disposal. This would mean that the $2n$ coordinates q_i and p_i can be transformed into some new Q_i and P_i by any functional relations we please. This, however, is not the case." For the non-singular systems, the necessary and sufficient condition that the transformation (148) to the new set of variables (Q_i, P_i) is a canonical transformation (and keeps the two formulation equivalent), is

$$\{Q_i, Q_k\}_{Q,P} = \{Q_i\left(p,q\right), Q_k\left(p,q\right)\}_{p,q} = 0,$$

$$\{P_i, P_k\}_{Q,P} = \{P_i\left(p,q\right), P_k\left(p,q\right)\}_{p,q} = 0, \tag{151}$$

$$\{Q_i, P_k\}_{Q,P} = \{Q_i\left(p,q\right), P_k\left(p,q\right)\}_{p,q} = \delta_{ik}.$$

The change of phase space variables (canonical transformations) for unconstrained Hamiltonians is an old and well established topic that can be found in many textbooks on Classical Mechanics. For constrained Hamiltonians the situation is different and even the number of papers (e.g. see [76, 77]) that discuss the general questions or particular examples (which are mainly final dimensional and artificial) of such changes is minuscule compared to the number of articles in which such changes are used without any analysis of their consequences. Such changes are especially common in Hamiltonian formulations of GR (in both Einstein [6] and Einstein-Cartan [78, 79, 80] forms).

The Dirac's generalization of the Hamiltonian formulation to constrained systems leads to the system of equations which is similar to (147):

$$q_i = \{q_i, H_T\}, \;\; p_i = \{p_i, H_T\},$$

where H_T also includes the primary constraints. We restrict our discussion to gauge invariant systems with only first-class constraints. For such systems, the conditions that should be made on possible changes of variables in phase space seems to be more restrictive. The reason for this is that gauge invariance depends on all of the first-class constraints and their PB algebra [10, 11, 12]. In addition, it is related to the total Hamiltonian of a particular model, in contrast to unconstrained systems where the conditions (151) are, in fact, independent of the Hamiltonian. For gauge invariant systems the change of variables (150) must preserve gauge invariance, i.e. gauge invariance derived in the (Q_i, P_i) variables, after using the inverse transformations, must produce the same result as in the original (q_i, p_i) variables. The set of phase space transformations (148) that preserves this property is the *equivalent set* or, as in non-singular case, we can call such changes canonical transformations.

In linearized versions of the Dirac formulation (see the previous Sections) and that of [4] which were considered in [37], both formulations, despite having different constraints and Hamiltonians, lead to the same gauge invariance. The relation between the two formulations was discussed and it was shown that they are related by canonical transformations (151), which is exactly the condition known to be needed for non-singular Hamiltonians. Despite there being far more complicated expressions for constraints and transformations, a similar relation (151) exists between the non-linearized formulation of Dirac and [30]; and these, as we have shown, also have the same gauge invariance. These examples demonstrate that the ordinary condition for the transformation to be canonical, which is known for unconstrained Hamiltonians is also correct for the general (constrained) case; i.e. it is a necessary condition as before. Yet, it is not sufficient and we shall demonstrate this fact by way of example.

Surprisingly, we have been unable to find, in the literature, either a discussion of the relations between the ADM and Dirac phase space variables or the transformations between them; which is strange as many authors presume their equivalence by calling this formulation "Dirac-ADM". In [81] the authors called the variables N, N^i and g_{km} "an equivalent set" which "is analytically convenient and geometrically more significant". In footnote 5 of [81] (appeared in 1959) it is stated that "The properties of these variables are discussed in detail in a forthcoming paper by C. Misner" that we have not been able to find. The convenience and significance are not our present concern, but the question of equivalence of the ADM and Dirac sets of variables is important. We are interested in the equivalence of gauge transformations in the two approaches, i.e. we must work in the full phase space. The

complete relation between these two sets of variables is not known, but there is one PB, $\left\{N(x), \Pi^{kl}(x')\right\}$, that can be easily checked. The space-space components of the metric tensor and corresponding momenta are the same in both formulations, i.e. $\Pi^{kl} = \Pi^{kl}\left(p^{kl}\right) = p^{kl}$. The Π^{kl} of ADM given by (141) is equivalent to p^{kl} of Dirac given by (18). It is sufficient to check using the Dirac variables $g_{\mu\nu}$, $p^{\mu\nu}$ the PB $\left\{N(x), \Pi^{kl}(x')\right\}$ which, if ADM variables are canonical, must give zero. Using the corresponding fundamental PBs (124-126) we obtain:

$$\left\{N(x), \Pi^{kl}(x')\right\} = \left\{\left(-g^{00}\right)^{-1/2}, p^{kl}\right\}_{g,p} = -\frac{\delta}{\delta g_{kl}}\left(-g^{00}\right)^{-1/2} = \frac{1}{2}\left(-g^{00}\right)^{-3/2} g^{0k} g^{0l} \neq 0. \tag{152}$$

Once again, we have the result that depends on Dirac's simplifying assumption (32): if we impose $g^{0k} = 0$, then the PB of (152) gives zero. In general this PB is not zero and the transformation from $(g_{\mu\nu}, p^{\mu\nu})$ to (N, Π), (N^i, Π_i), and $\left(g_{km}, \Pi^{km}\right)$ is not canonical. One PB is enough to show this, irrespective of the results we might obtain for the PBs among the other phase space variables.

We take a note that equation (152) gives zero in the static coordinate system, but in this case the corresponding components, g_{0k}, and their conjugate momenta have to be dropped out of the formalism from the beginning, as in the case of the Hamiltonian formulation in the Schwarzschild metric [39] where only four components of the metric tensor are left in the Lagrangian before passing to the Hamiltonian.

This simple calculation, (152), allows us to conclude that the two Hamiltonians of Dirac and ADM are not related by a canonical transformation and the respective failure of the ADM variables and the Hamiltonian to produce a diffeomorphism transformation is a manifestation of this non-equivalence. Moreover, (152) shows that the ADM variables are not the canonical variables of GR. The converse statement is also true and the metric tensor is not a canonical variable of the ADM formulation. The ADM formulation might be considered as a model (geometrodynamics or ADM gravity) without any reference to Einstein GR, but in this case a "correspondence" between the two transformations (134-137) is, in fact, meaningless. The transformations that follow from ADM are given by (132), (138), and (142) and in the absence of canonicity we cannot return to the transformations of the metric tensor. There is another characteristic that supports the loss of connection with the original

variables: it is impossible to find any redefinition of Π and Π_k in terms of Dirac's phase space variables (whether they satisfy (151) or not) that can transform his total Hamiltonian (116) into the ADM total Hamiltonian (115).

In general, no algorithm exists for finding a canonical transformation but the canonicity of a given transformation can be checked. There are no canonical transformations for Hamiltonians that involve only a change of generalized coordinates, as this change must be accompanied by transformations of the momenta that can be found by using the following procedure [74]. If the transformations of the generalized coordinates (fields) are given, one can find the corresponding transformations of the momenta that will guarantee that the new coordinates and momenta are canonical and satisfy (151) using the relation

$$p_i \delta q_i = P_i \delta Q_i.$$

Let us see what we can obtain from this relation for the ADM change of variables

$$p^{\alpha\beta}\delta g_{\alpha\beta} = \Pi \delta N + \Pi_k \delta N^k + \Pi^{km}\delta g_{km}.$$

By performing the variations $\delta Q = \frac{\delta Q}{\delta g_{\alpha\beta}}\delta g_{\alpha\beta}$ using (119-120), we find that

$$p^{\alpha\beta} = \Pi \frac{\delta N}{\delta g_{\alpha\beta}} + \Pi_k \frac{\delta N^k}{\delta g_{\alpha\beta}} + \Pi^{km}\frac{\delta g_{km}}{\delta g_{\alpha\beta}}$$

which gives

$$p^{00} = -\Pi \frac{1}{2}\left(-g^{00}\right)^{1/2}, \tag{153}$$

$$p^{0m} = \Pi \frac{1}{2}\left(-g^{00}\right)^{-1/2} g^{0m} + \Pi_k \frac{1}{2} e^{km} \tag{154}$$

and

$$p^{pq} = -\Pi \frac{1}{2}\left(-g^{00}\right)^{-3/2} g^{0p} g^{0q} + \Pi_k \frac{1}{2}\left(\frac{g^{0p}}{g^{00}} e^{kq} + \frac{g^{0q}}{g^{00}} e^{kp}\right) + \Pi^{pq}. \tag{155}$$

Now solving for the Π's:

$$\Pi = -2\left(-g^{00}\right)^{-1/2} p^{00}, \tag{156}$$

$$\Pi_n = 2g_{mn}p^{0m} + 2g_{0n}p^{00}, \tag{157}$$

$$\Pi^{pq} = p^{pq} + \frac{g^{0q}}{g^{00}}\frac{g^{0p}}{g^{00}}p^{00} - \frac{g^{0p}}{g^{00}}p^{0q} - \frac{g^{0q}}{g^{00}}p^{0p}. \tag{158}$$

Note that, to have $\Pi^{pq} = p^{pq}$, as in the case of the Dirac and ADM Hamiltonians, we again must impose the condition $g^{0k} = 0$.

Only if equations (156-158) are taken together with the relations for the generalized coordinates (119-120), are the transformations canonical. It is not difficult to check that the canonical properties of PBs are preserved, and as an example, the PB that we considered in (152) gives

$$\{N(x), \Pi^{pq}(x')\} = \left\{ \left(-g^{00}\right)^{-1/2}, p^{pq} + \frac{g^{0q}}{g^{00}}\frac{g^{0p}}{g^{00}}p^{00} - \frac{g^{0p}}{g^{00}}p^{0q} - \frac{g^{0q}}{g^{00}}p^{0p} \right\}_{g,p} = 0,$$

as it should for variables that are connected by canonical transformations. For *non-singular* Lagrangians and their corresponding Hamiltonians, the ADM change of variables accompanied by the change of momenta (153-155) would be sufficient to obtain the new set of canonical variables. If GR were a non-singular theory, these transformations would guarantee equivalence between the two formulations; but for the constrained Hamiltonian this is not the case. If the canonical transformation of (122), (153-155) are performed in the Dirac Hamiltonian, we will not obtain the ADM Hamiltonian, and we will not obtain a consistent result. In particular, for a description of a constrained system, the total Hamiltonian, H_T, is important as it includes all the primary constraints. In the Dirac formulation, these are (22):

$$g_{00,0}p^{00} + 2g_{0k,0}p^{0k}. \tag{159}$$

Substitution of (122), (153-155) into this equation gives

$$N_{,0}\Pi + N^m_{,0}\Pi_m + g_{kj,0}\left(\frac{1}{2}\Pi\frac{N^kN^j}{N} + N^j\Pi_m e^{mk}\right)$$

which is nonsensical. The first two terms are equivalent to (159) but an extra term appears, which is zero only if $N^k = 0$. The space-space velocities have already been eliminated in favour of their corresponding momenta but now they reappear and it is not clear what to

do with them at this stage. If we treat them on the same footing as the rest of the terms with time derivatives, we must specify their coefficients as primary constraints, which would then give a total of ten primary constraints. The same change of variables in the canonical part of the Hamiltonian will give contributions that are quadratic in all momenta. Without further analysis we see that this will probably lead to some contradictions, as it is clear that the constraint structure of the Hamiltonian is changed and one would expect second-class constraints, etc.

The foregoing example, (119-120) and (156-158), clearly demonstrates that the condition (151), which is necessary and sufficient for the transformation to be canonical in the case of non-singular Lagrangians, is not a sufficient for singular Lagrangians.

Actually, for the ADM change of variables, the non-canonical nature of the transformations is immediately clear and this is not even related to a singular structure of the Lagrangian of GR. The possible existence of additional restrictions beyond (151) is understandable, and for the Hamiltonian with first-class constraints it can be expected. Gauge transformations are derived from the first-class constraints and the whole PB algebra of constraints plays a key role in this derivation. It is not enough to have the same number of constraints in the two formulations. The ADM change of variables keeps the same number of constraints as the Dirac formulation; but the PB algebra of constraints is affected. The simplest example is a PB among primary and secondary constraints, which are zero in the ADM case and proportional to true constraints in the case of the Dirac Hamiltonian (63). From a mathematical point of view (we have already spent enough time on the interpretational aspects) the ADM formulation is just the result of a non-canonical change of variables in Dirac's Hamiltonian of GR and if Dirac's formulation allows one to derive the diffeomorphism transformations, then the ADM formulation, because of this non-canonical change of variables, does not allow one to restore either the full diffeomorphism invariance for all components of $g_{\mu\nu}$ or even for its spatial part (as it is usually claimed) without a non-covariant and field-dependent redefinition of gauge parameters. In addition, we observed that some equations in the ADM formulation are true only if $g_{0k} = 0$. So the ADM change of variables is somehow also related to a static coordinate system, but in a strange and obscure way: g_{0k} is not zero at the outset, but later should be set equal zero so that some subsequent relations are made valid.

We conclude that the Dirac Hamiltonian of GR was obtained by following the "rule

of procedure" and, because of this, it is canonical already and the use of the adjective "canonical" is a tautology. It is not equivalent to the ADM Hamiltonian, which, as we have demonstrated, is the result of a non-canonical change of variables. The ADM formulation is obtained by abandoning the "rule of procedure" and, consequently, only by a canonization can it be ironically called the "canonical formulation of GR".

One can argue that the ADM Hamiltonian is not obtained from the Dirac Hamiltonian. There are no references in [6] to Dirac's article [5], the only reference where a derivation of Dirac's Hamiltonian has been considered. In their culminating paper [6] and in 13 preceding articles it is mentioned only once in [72] and in a different context. However, according to both Dirac and ADM, their respective Hamiltonians are derived from the same theory - GR. In both cases some modifications of the EH action were performed. In Dirac's case these modifications are explicitly stated, in the ADM case it is more difficult to trace what has been done. Let us start from the Dirac Lagrangian. We have discussed above how, in the course of a Hamiltonian analysis that follows Dirac, the possibility of additional integrations by parts appear when the Hamiltonian is to be expressed as a linear combination of secondary constraints (see our discussion after (55)). The modified Lagrangian can be written in the following form (up to total derivatives as in (54))

$$L_{Dirac} = \frac{1}{4}\sqrt{\det g_{km}}\left(-g^{00}\right)^{1/2} E^{rsab}\left[\left(-g_{ap}\frac{g^{0p}}{g^{00}}\right)_{,b} + \left(-g_{bp}\frac{g^{0p}}{g^{00}}\right)_{,a} - g_{ab,0} + \frac{g^{0k}}{g^{00}}\left(g_{ak,b} + g_{bk,a} - g_{ab,k}\right)\right]$$

$$\times\left[\left(-g_{rq}\frac{g^{0q}}{g^{00}}\right)_{,s} + \left(-g_{sq}\frac{g^{0q}}{g^{00}}\right)_{,r} - g_{rs,0} + \frac{g^{0m}}{g^{00}}\left(g_{rm,s} + g_{sm,r} - g_{rs,m}\right)\right]$$

$$+\sqrt{\det g_{km}}\left(-g^{00}\right)^{-1/2}\left[g_{mn,kt}E^{mnkt} + \frac{1}{4}g_{mn,k}g_{pq,t}\left(E^{mnpq}e^{kt} - 2E^{ktpn}e^{mq} - 4E^{pqnt}e^{mk}\right)\right] \tag{160}$$

By performing the change of variables of (122) in (160), we obtain the ADM Lagrangian

$$L_{ADM} = \frac{1}{4}\sqrt{\det g_{km}}\frac{1}{N}E^{rsab}\left[\left(g_{ap}N^p\right)_{,b} + \left(g_{bp}N^p\right)_{,a} - g_{ab,0} - N^k\left(g_{ak,b} + g_{bk,a} - g_{ab,k}\right)\right]$$

$$\times\left[\left(g_{rq}N^q\right)_{,s} + \left(g_{sq}N^q\right)_{,r} - g_{rs,0} - N^m\left(g_{rm,s} + g_{sm,r} - g_{rs,m}\right)\right]$$

$$+\sqrt{\det g_{km}}N\left[g_{mn,kt}E^{mnkt}+\frac{1}{4}g_{mn,k}g_{pq,t}\left(E^{mnpq}e^{kt}-2E^{ktpn}e^{mq}-4E^{pqnt}e^{mk}\right)\right]. \quad (161)$$

Or, by using the intrinsic and extrinsic curvatures, ${}^3R_{rs}$ and K_{rs}, respectively, it can be written in more familiar form [15, 71]

$$L_{ADM}=\sqrt{\det g_{km}}N\left(E^{rsab}K_{rs}K_{ab}+{}^3R\right) \quad (162)$$

with

$$K_{rs}=\frac{1}{2N}\left(N_{r|s}+N_{s|r}-g_{rs,0}\right)$$

where, as before, "|" means covariant derivative with respect to three dimensional metric ($N_{r|s}=N_{r,s}-\Gamma^k_{rs}N_k$).

Now using (161) or (162) we can easily obtain the ADM Hamiltonian, which is the same as (115). However, following such a detour we cannot avoid the question of whether the ADM variables are canonical. The transformations of the metric tensor derived from the Hamiltonian formulation of the Dirac Lagrangian (160) and from the Hamiltonian formulation of the ADM Lagrangian (161) are different, the destination is changed. This detour is a wrong turn or perhaps a dead end road.

We combine the results of these two formulations into a compact visual form[24]:

$$\begin{array}{ccc}
L_{Dirac}(q) & \overset{q=q(Q)}{\Longrightarrow} & L_{ADM}(Q) \\
\Downarrow & & \Downarrow \\
H_{Dirac}(p,q) & \neq & H_{ADM}(P,Q) \\
\Downarrow & & \Downarrow \\
 & & \delta_{ADM}Q \\
 & & \Downarrow Q(q) \\
\delta_{diff}q & \neq & \delta_{ADM}q
\end{array} \quad (163)$$

It is reasonable to expect that if by a change of variables $\overset{q=q(Q)}{\Longrightarrow}$ we can obtain a new equivalent Lagrangian and find the corresponding Hamiltonian, then the two Hamiltonians (in

[24] This 'pictorial visualization' is based on results of calculations, not the other way around.

both the new and the original variables) should also be equivalent, be related to each other by a canonical transformation, and necessarily lead to the same gauge invariance. If the Hamiltonians are not related by a canonical transformation, then the two Lagrangians are not equivalent. This is the natural conclusion that one can make. In particular, application of the Lagrangian methods used by Samanta [26] to derive the diffeomorphism invariance of GR, when applied to the ADM Lagrangian, should not give the diffeomorphism transformations [82] but rather the same transformations as obtained in its Hamiltonian treatment in [19]. This is a consequence of the well-known equivalence of the Lagrangian and Hamiltonian formulations for any system, either non-singular or singular [17, 83]: the "vertical" equivalence of (163). If a "horizontal" equivalence is broken, either for the two Hamiltonians or for the gauge transformations, then it is broken everywhere, including at the Lagrangian level.

The Hamiltonian formulations of the linearized versions of the Dirac and gamma-gamma Lagrangians lead to the same algebra of constraints and gauge transformations (although the Hamiltonians themselves and the constraints are different). But the two Hamiltonians are related by a canonical transformation [37]. Such transformations also exist in the case of the corresponding full Dirac and the gamma-gamma formulations of GR [42]. The equivalence of the PB algebra of constraints can be easily seen by comparing [30] to the results of Section II. The explicit canonical transformation and its effect on constraints, their algebra and structure functions will be given in [42].

The main subject of this article is the Hamiltonian formulation of GR in second-order form and, in particular, a comparison of the formulations related by a change of variables. However, we think that some comments on similar changes made to its Lagrangian should be given.

It is often stated that in a Lagrangian formulation of a model any field redefinition is legitimate provided it is invertible (i.e. it has a non-zero Jacobian). For singular Lagrangians, and especially gauge invariant ones, this is obviously not a sufficient condition. One additional restriction in gauge invariant cases is the preservation of the rank of Hessian as this gives us the number of gauge parameters (if all constraints are first-class). We cannot have two equivalent formulations if they have a different number of gauge parameters, and we cannot, by a change of variables, eliminate some gauge invariance or create a new gauge invariance. If we were to make such a change, we can of course, treat the new Lagrangian as

some different model, but we cannot relate it to the original one as any connection with the original theory is lost. Obviously these two conditions are necessary, but not sufficient. The ADM change of variables satisfies them both, but leads to different gauge transformations (compare [30] and Section III versus [19] and (134-137)). A change of variables in singular (in particular, gauge invariant) Lagrangians is a much more restrictive procedure if one intends to preserve its equivalence with the initial formulation. One way is to rely on the Hamiltonian method and check if the new variables are canonical and the two total Hamiltonians are equivalent, including the primary constraints. We must also check whether the entire algebra of constraints is equivalent, as this algebra is responsible for the gauge transformations. This can be considered as a confirmation of Dirac's statement [7] "I feel that there will always be something missing from them [non-Hamiltonian methods] which we can only get by working from a Hamiltonian, or maybe from some generalization of the concept of a Hamiltonian". However, we think that some criteria for the equivalence between two sets of variables for singular Lagrangians can be formulated at the pure Lagrangian level. At the Lagrangian level, a gauge invariance is related to the existence of gauge identities [17] and an inappropriate change of fields can modify or even destroy them. This echoes the conclusion of Isham and Kuchar [73] that "... space-time diffeomorphism has somehow got lost in making the transition from the Hilbert action to the Dirac-ADM action"[25].

There is another indication of the incorrectness of the ADM change of variables at the Lagrangian level that comes from Numerical Relativity. In almost all methods of numerical integration of the Einstein equations the starting point is the ADM 3+1 decomposition. So at the outset the Einstein equations are replaced by the ADM equations [6]. It was shown that, in contrast to the Einstein equations which are strongly (strictly) hyperbolic (SH) [85], the ADM equations are weakly hyperbolic (WH) (e.g., see [86, 87, 88]). *This change in the type of equations* is related to the different constraint structure and different transformations derived from a non-equivalent Hamiltonian. From a computational point of view there is a fundamental difference between SH and WH systems of PDEs: the former are well-posed and convergent, whereas the latter are not well-posed and divergent [89]. There is numerical evidence that ADM-based algorithms are unstable. As is indicated in [90]: "The common

[25] We have demonstrated the inequivalence of the Dirac and ADM formulations. In [73] the authors discussed geometrodynamics which uses ADM variables and their statements should be applied to the ADM action only.

lore these days is, however, that the standard Arnowitt-Deser-Misner (ADM) formulation is the one which most easily suffers instabilities". Or in [89]: "It took several years to realize that such instabilities were not associated with the numerical algorithms but rather with the mathematical structure of the ADMY [ADM [6] and York [91]] system itself". This is not just an additional indication of the incorrectness of the ADM change of variables, but also a demonstration that even as a model, geometrodynamics is an *ill defined theory.*

In general, the change in the type of equations, from SH to WH, or a change of "level of hyperbolicity" [92], is an indication that in the process of transforming from the Einstein to the ADM equations, some "damage" was done (as there is no longer a complete set of eigenvectors associated with the characteristic matrix [93]). The proposed "cure" [90] of such "damage" in most approaches lies in a modification (or "adjusting") of the ADM equations by adding terms involving constraints [94] (trying to restore what has been lost) or by using different choices of the lapse and shift functions [90] to give the ADM system of equations well-posedness (or "quasi well-posedness" [90]), rather than returning to the original Einstein equations and constraints that preserve diffeomorphism invariance.

The main point of this relatively long Section is not to prove that the ADM variables are not canonical variables for GR (as is shown by the one simple PB (152)), but to demonstrate and discuss the restrictive conditions that must be made on change of variables in any Hamiltonian formulation of a singular Lagrangian, using GR as an example. A blind change of variables in singular systems without performing a thorough analysis and without developing mathematical criteria for such changes can lead to a wrong result. All new variables that are introduced in such cases, regardless of their physical or geometrical meaning, regardless of what new names were given to reflect their interpretation, or after whom new variables were named, must be carefully analyzed if one wants to keep all the properties of the original theory intact or, in other words, if one intends to study the original theory and not a substitute, which can be ill defined, even as unrelated to the original theory model. One should under no circumstances attribute to the original theory the contradictions or problems that arise after such inappropriate changes are made; and never project any novel result or discovery obtained by abandoning a "regular and uniform rule of procedure" into the original theory or to Nature Herself.

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